2^{nd}-order topology and supersymmetry in 2D-topological insulators

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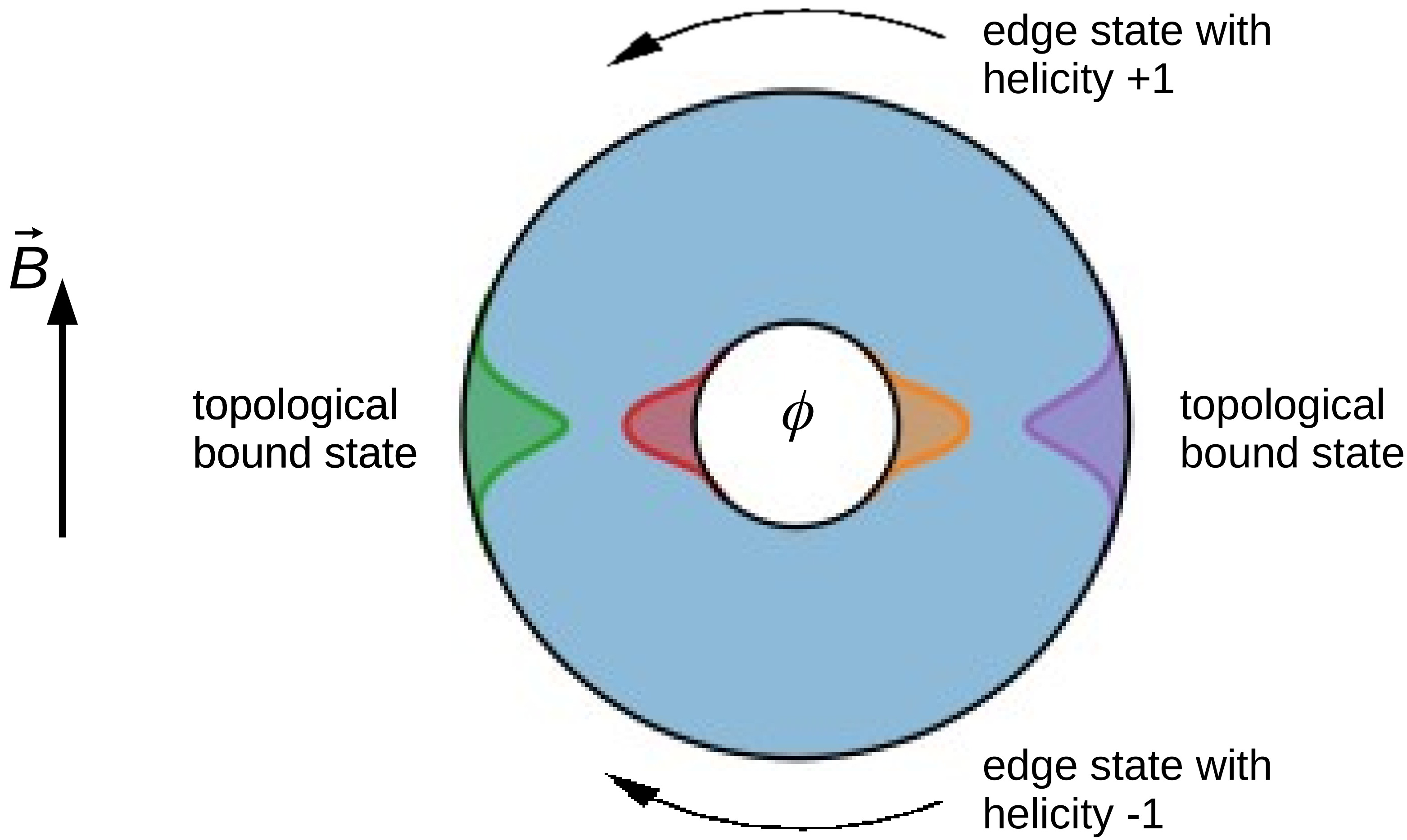
- Second-order topology in two dimensions: BHZ model + Zeeman term
- Half-integer flux through a hole in the system: Supersymmetry (SUSY)
  → induced by **anticommuting inversion and mirror symmetry**
- Topological protection of zero-energy states by **SUSY** + **chiral symmetry**
- Universal low-energy theory in terms of an effective surface Hamiltonian
  → realization of the whole class of **periodic Witten models in 1D**
- Topological engineering with hole states

2nd-order topology:

bulk gap → two counterpropagating helical edge modes at the boundary of the system

surface gap → topological bound states at certain positions of the surface where the mass term changes sign



supersymmetry at half-integer flux: $f = \frac{\phi}{\phi_0} = 1/2$

Minimal model for second-order TI in 2D: BHZ + Zeeman

Langbehn et al., PRL '17
Geier et al., PRB '18
Khalaf, PRB '18
Ren et al., PRL '20

BHZ:
- band inversion: $\sigma_z\left(\frac{p^2}{2m} - \delta\right)$
- spin-orbit: $\alpha\,\sigma_x \vec{p}\,\vec{s}$

Zeeman: $E_Z \hat{\vec{B}}\,\vec{s}$

Strong spin-orbit: $|\delta|, E_Z < 2E_{so} = m\alpha^2$

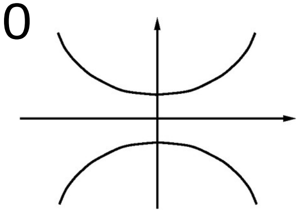
Bulk gap: $\Delta_{bulk} = |\delta| - E_Z$

$$\tilde{\delta} = \frac{\delta}{E_{so}} \qquad \tilde{E}_Z = \frac{E_Z}{E_{so}} = \frac{1}{\tilde{l}_B^2}$$

$\tilde{l}_B = \frac{l_B}{\lambda_{so}}$ magnetic length

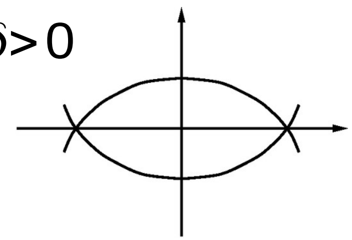
$\lambda_{so} = \frac{1}{\alpha m}$ spin-orbit length

$\delta < 0$

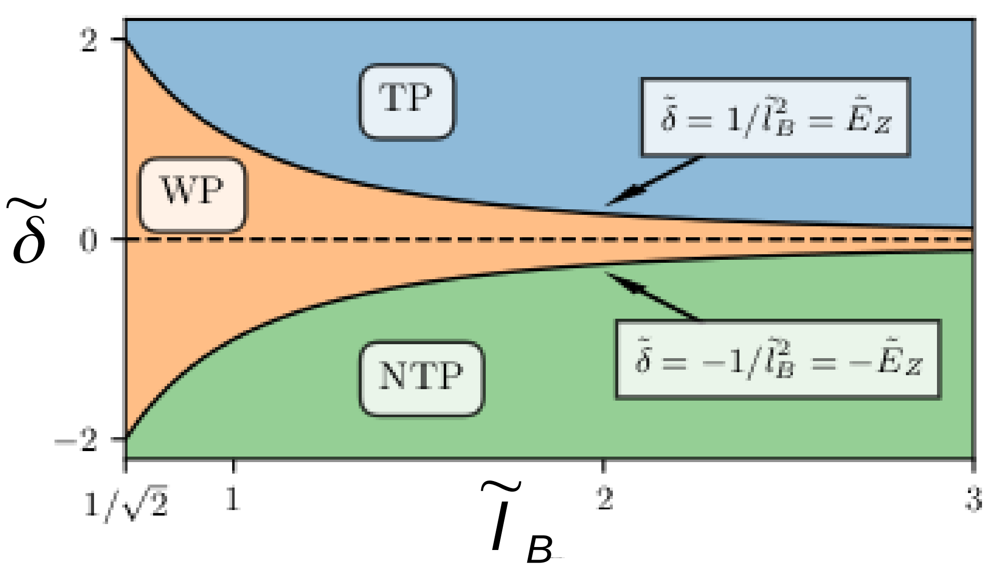


trivial

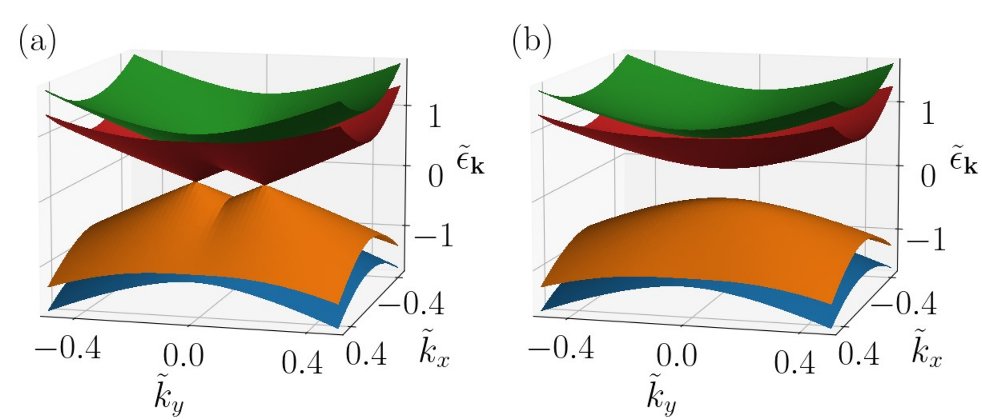
$\delta > 0$



topological

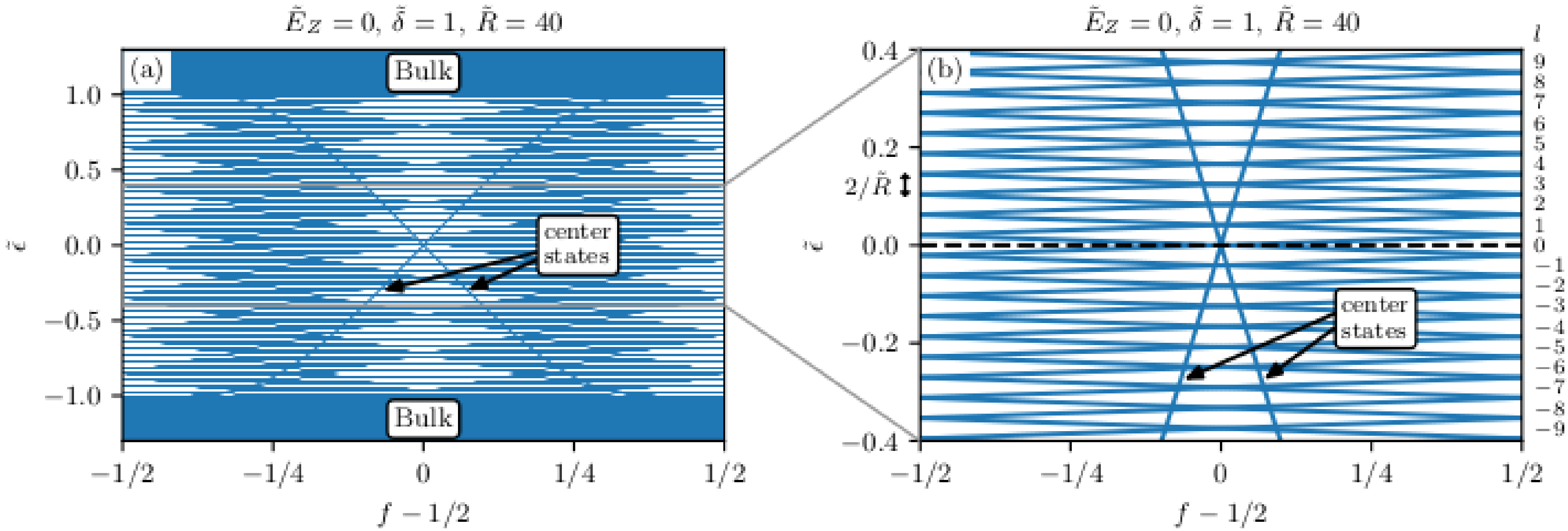


Weyl phase (WP)   Topological phase (TP)

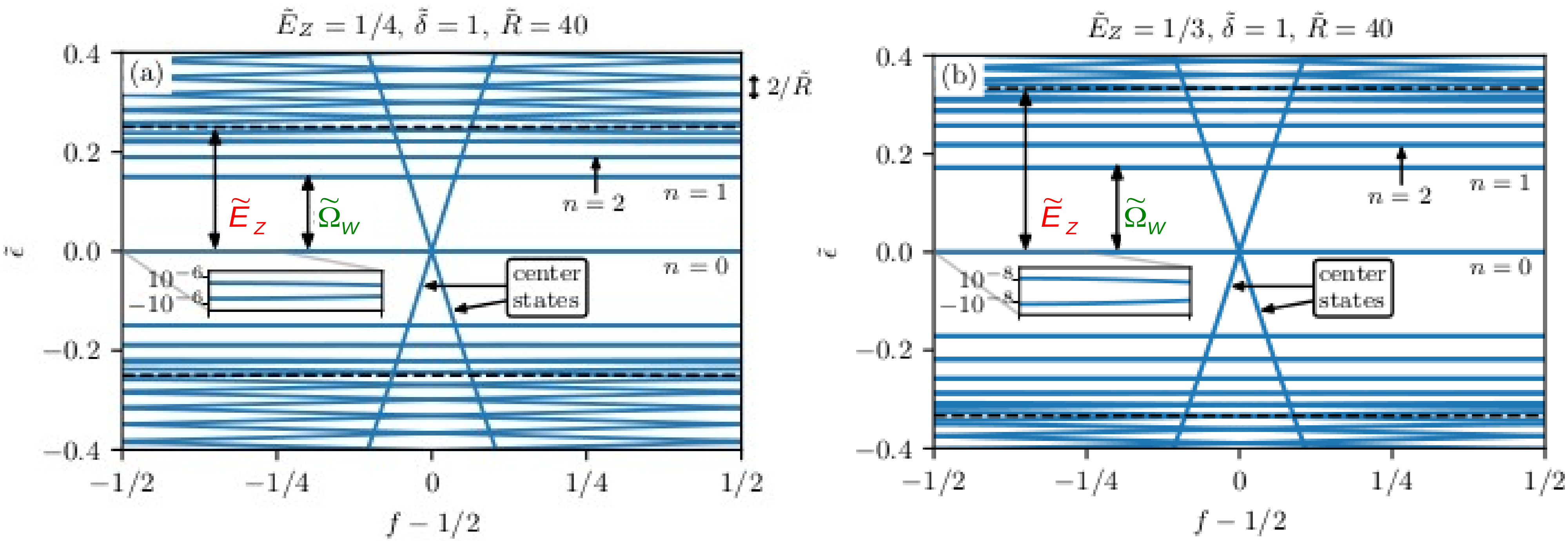


Flux dependence of counter-propagating edge states: $E_Z = 0$

disc



Surface gap opening at finite Zeeman field: $E_Z \neq 0$



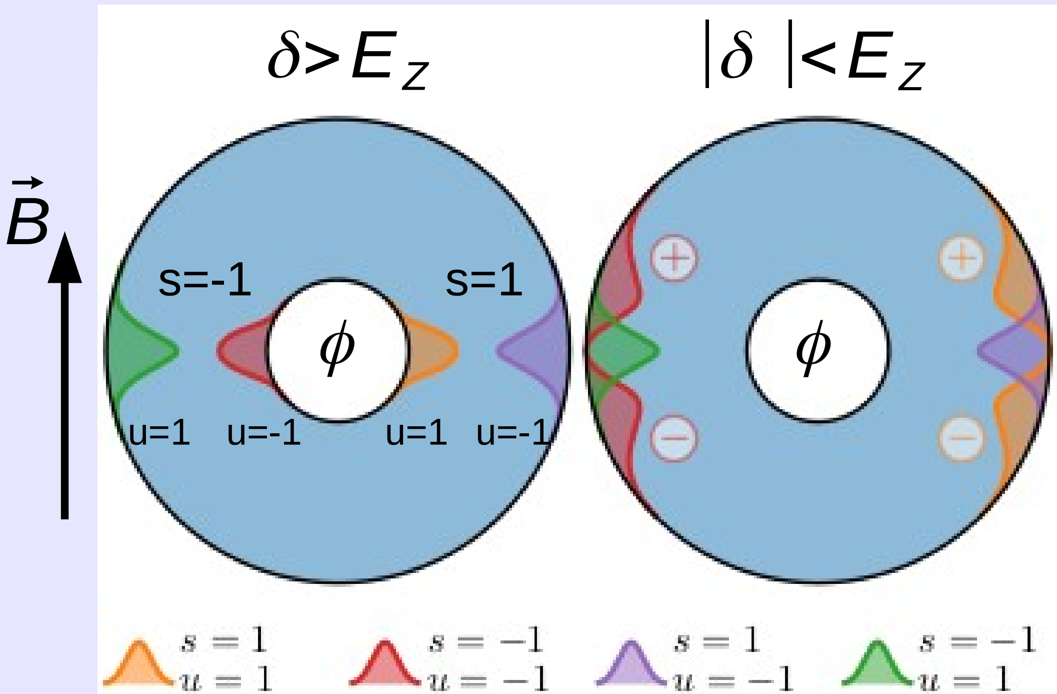
Topological states and phase diagram for a Corbino disc

Corbino disc:

AB-flux : $f = \frac{\phi}{\phi_0}$

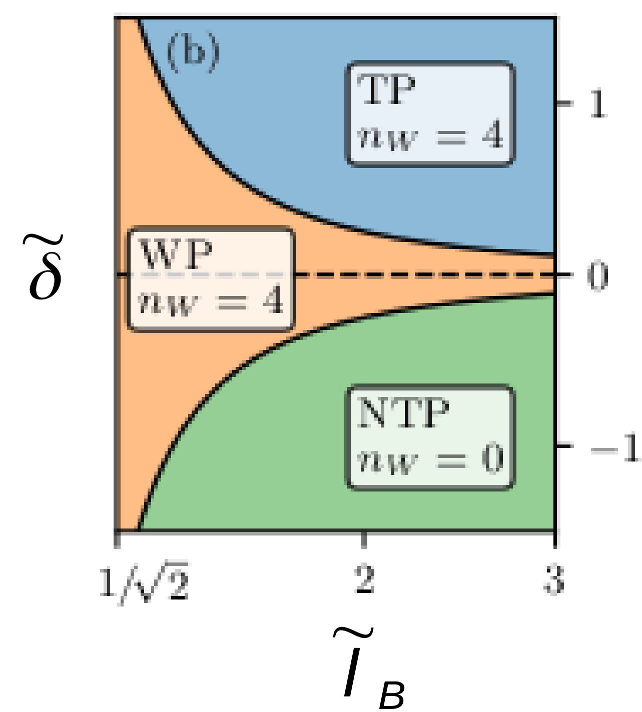
SUSY : $f = \frac{1}{2}$

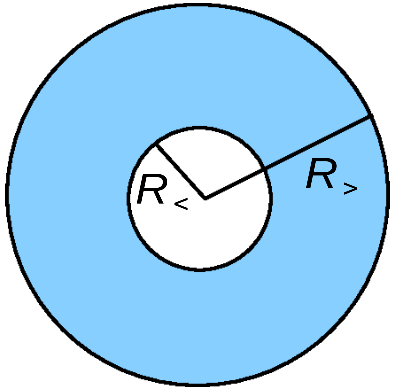
Topological phase   Weyl phase



s=±1 → chiral symmetry   u=±1 → SUSY

n_W → Witten index
→ # zero-energy states



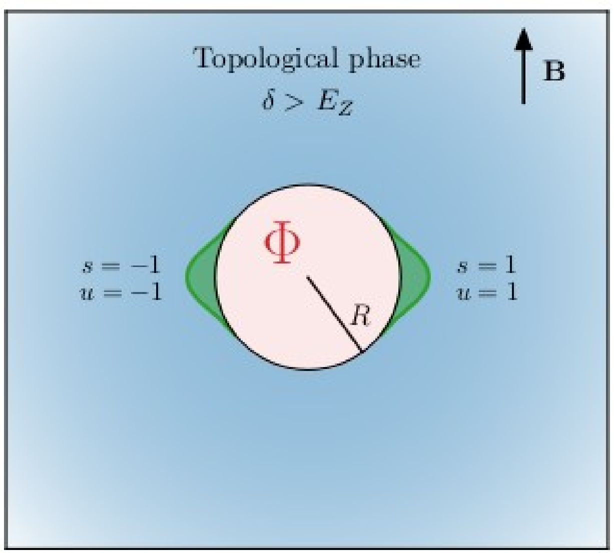


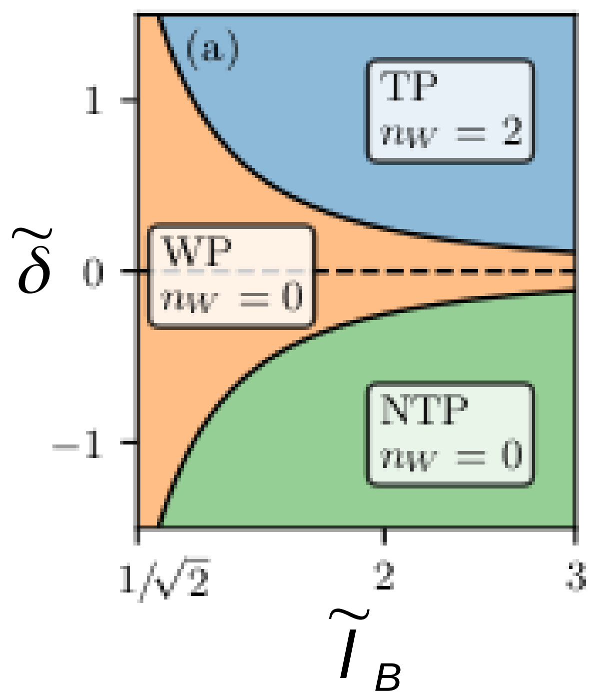
$$|R_> - R_<| \gg \xi_n \sim \lambda_{so}$$

$\Rightarrow$ exponentially small splitting
of states localized at different surfaces

$\Rightarrow$ topological states have
exponentially small energy

Hole in an infinite system: topological states are exactly at zero energy



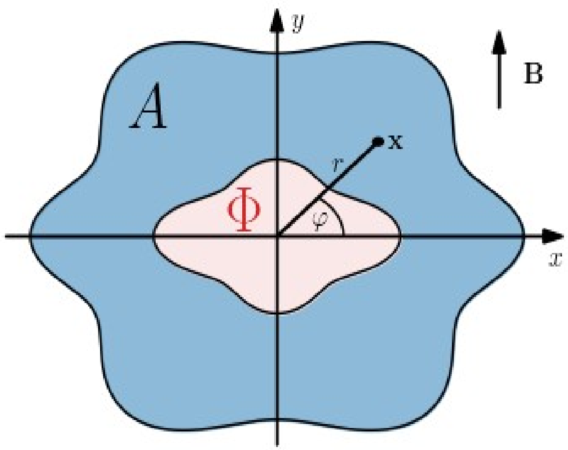


Supersymmetry at half-integer flux: f = 1/2

Chiral: $S = \sigma_x s_z$

Inversion: $\Pi = P_{\underline{x}} \sigma_z$

Mirror: $U = e^{-i\varphi/2} P_\varphi \sigma_z s_y e^{i\varphi/2}$
(→SUSY) $= e^{-i\varphi} P_\varphi \sigma_z s_y$



mirror symmetric area A

$f = \frac{\phi}{\phi_0} = \frac{1}{2}$

$\underline{U\Pi = -\Pi U}$: $H|\psi\rangle = \epsilon|\psi\rangle$ $U|\psi\rangle = \pm|\psi\rangle$ $U^2 = 1$

$\Rightarrow \langle\psi|\Pi|\psi\rangle = \langle\psi|U\Pi U|\psi\rangle = -\langle\psi|\Pi U^2|\psi\rangle = -\langle\psi|\Pi|\psi\rangle = 0$

Π unitary $\Rightarrow$ $\Pi|\psi\rangle \neq 0$ another eigenstate with the same energy

$\Rightarrow$ **exact 2-fold degeneracy of all eigenstates**

$|\psi\rangle$ and $\Pi|\psi\rangle$ → degenerate

- **SUSY**: pair of zero-energy states can not split
- **Chiral symmetry**: pair of zero-energy states can not shift

$\Rightarrow$ **topological protection**

n=1 SUSY representation for Witten Hamiltonian:

$$H_W = H^2$$

$$Q = H\Pi = Q^\dagger$$

hermitian supercharge operator

$$\mathrm{H}_W = Q^2$$

$$QU = -UQ$$

$$U^2 = 1$$ involution

one hermitian supercharge operator Q
+ involution U

$$[S, \mathrm{H}_W] = [S, U] = [S, Q] = 0$$

⇒ construction possible for each chiral sector separately

For each chiral sector: $|\psi\rangle$ and $Q|\psi\rangle$ → degenerate SUSY partners if

$$H_W|\psi\rangle = Q^2|\psi\rangle = \lambda|\psi\rangle$$

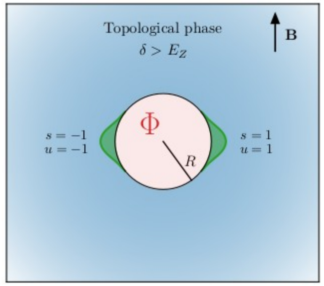
$$Q|\psi\rangle \neq 0 \iff \lambda > 0$$

But: Q is not unitary ⇒ $Q|\psi^{(0)}\rangle = 0$ is possible for the ground state of H_W

⇒ **single state with $\lambda = 0$ is possible in each chiral sector!**

→ distinguishes between broken/unbroken SUSY

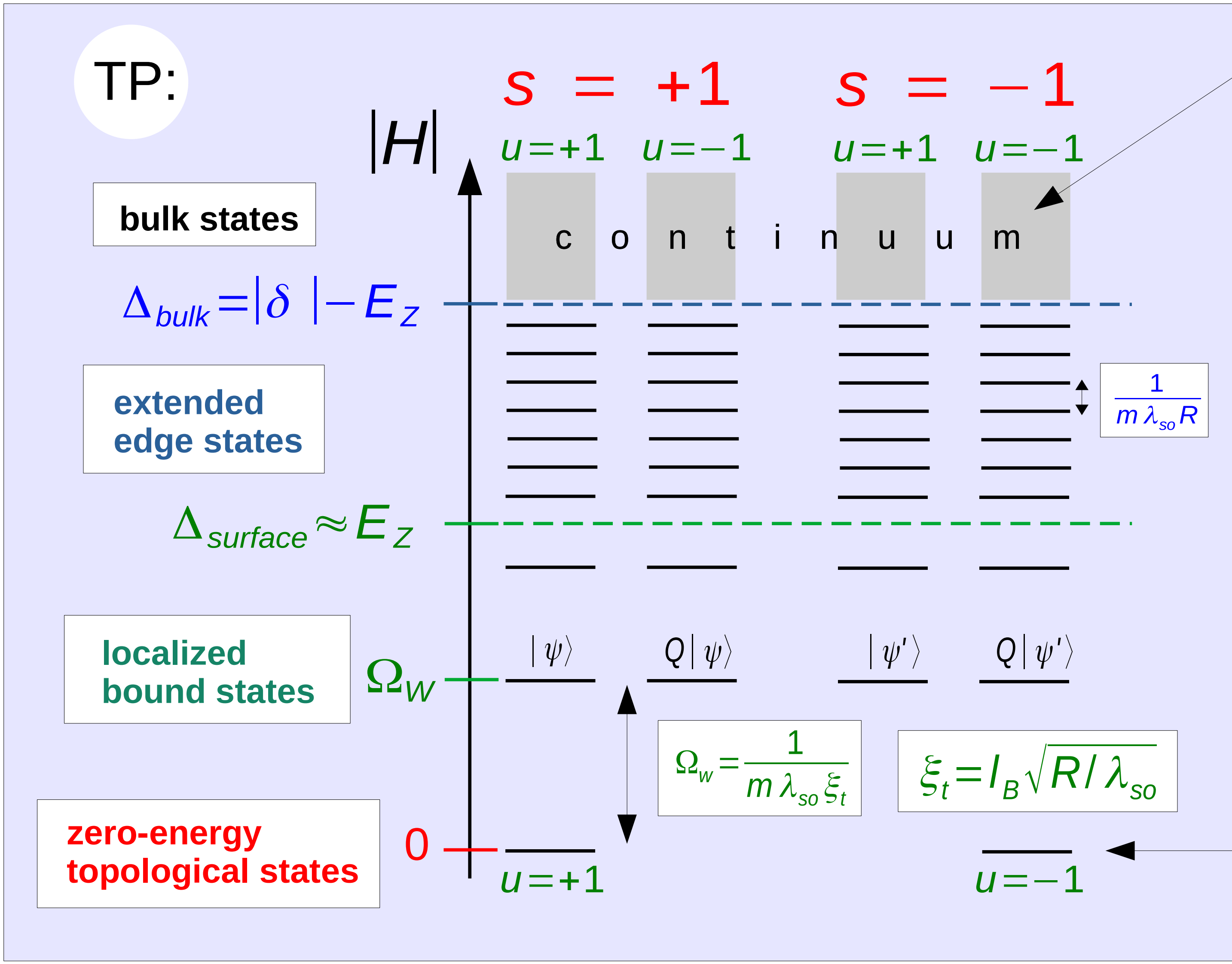
Hole in an infinite system:



$$\mathrm{H}_W = H^2$$

Witten model

$$[S, \mathrm{H}_W] = [U, \mathrm{H}_W] = 0$$
$$[S, U] = 0$$



4-fold degeneracy of all bulk states

exact SUSY spectrum of all states in each chiral sector

TP:

unbroken SUSY

WP + NTP:

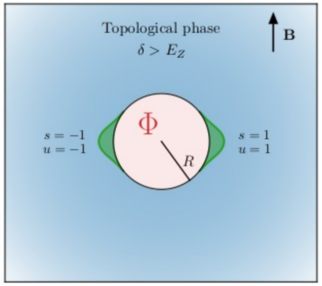
broken SUSY

exist only in TP

► states localized at the boundary → can be calculated for a smooth surface from an effective surface Hamiltonian

Effective surface Hamiltonian

hole system



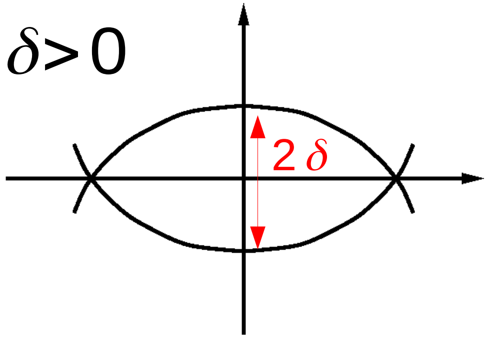
<u>Separation in normal and tangential part:</u>

$$\lambda_{so} \sim \xi_n \ll \xi_t \sim l_B \sqrt{R/\lambda_{so}} \ll R$$

<u>Normal part:</u> radial part of band inversion and spin-orbit (spinor degrees are transformed)

$$H_{normal} = \sigma_x [(-\frac{1}{2m} \partial_r^2 - \delta) s_z + 2 i \alpha \partial_r s_y]$$

$\delta > 0$



zero-energy
hole edge state with

$$s_x = -1$$

→ huge degeneracy in angular space

<u>Zeeman term:</u>

$$E_Z \sigma_y (s_x \sin \varphi + s_y \cos \varphi)$$

normal component opens surface gap

tangential component
→ small effect for $E_Z \ll \Delta_{bulk}$

Effective surface Hamiltonian:

(half-integer flux)

$$H^{eff}_{surface} = -\frac{\alpha}{R}\sigma_x(-i\partial_\varphi) - \sigma_y E_z \sin\varphi$$

angular part of spin-orbit

normal component of Zeeman

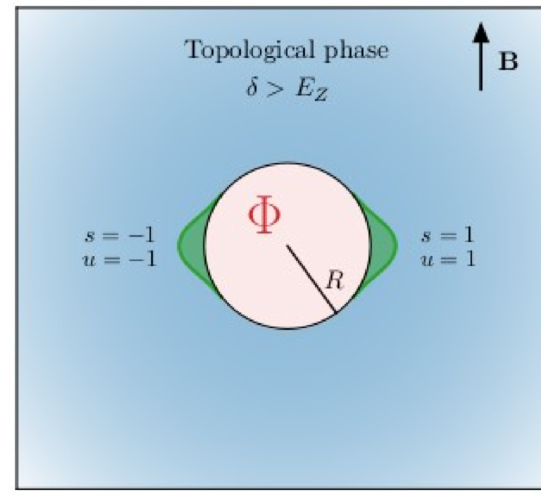
Witten model:

$$H_W = (H^{eff}_{surface})^2 = -\frac{\alpha^2}{R^2}\partial^2_\varphi + V_W^{-\sigma_z}(\varphi)$$

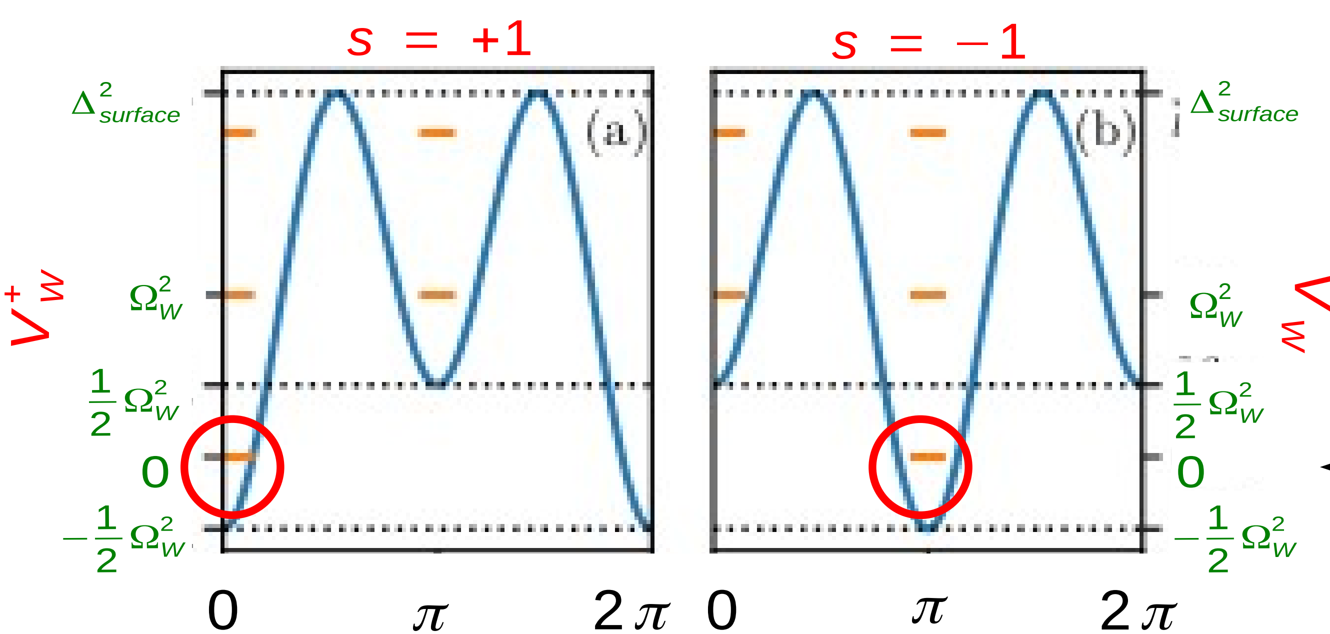
$s = -\sigma_z$ chiral symmetry

$$V_W^{\pm}(s_t) = E_Z^2 \sin^2\varphi \mp \frac{\alpha}{R} E_Z \cos\varphi$$

double sine potential



Trapping of bound states in effective surface potentials:



SUSY spectrum for each chiral sector

topological states

$$\Omega_w = \frac{1}{m\lambda_{so}\xi_t} \qquad \xi_t = l_B\sqrt{R/\lambda_{so}}$$

Arbitrary smooth surface: curvature radius $R \gg \xi_n \sim \lambda_{so}$

$s_t \rightarrow$ line element along the surface

Supersymmetric Dirac model: (for any mirror symmetric surface)

$$H^{eff}_{surface} = \alpha \sigma_x (-i \partial_{s_t}) + \sigma_y (E_Z)_{normal}(s_t)$$

Witten model:

$$H_W = (H^{eff}_{surface})^2 = -\alpha^2 \partial^2_{s_t} + V_W^{-\sigma_z}(s_t)$$

$s = -\sigma_z$

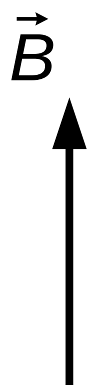
chiral symmetry

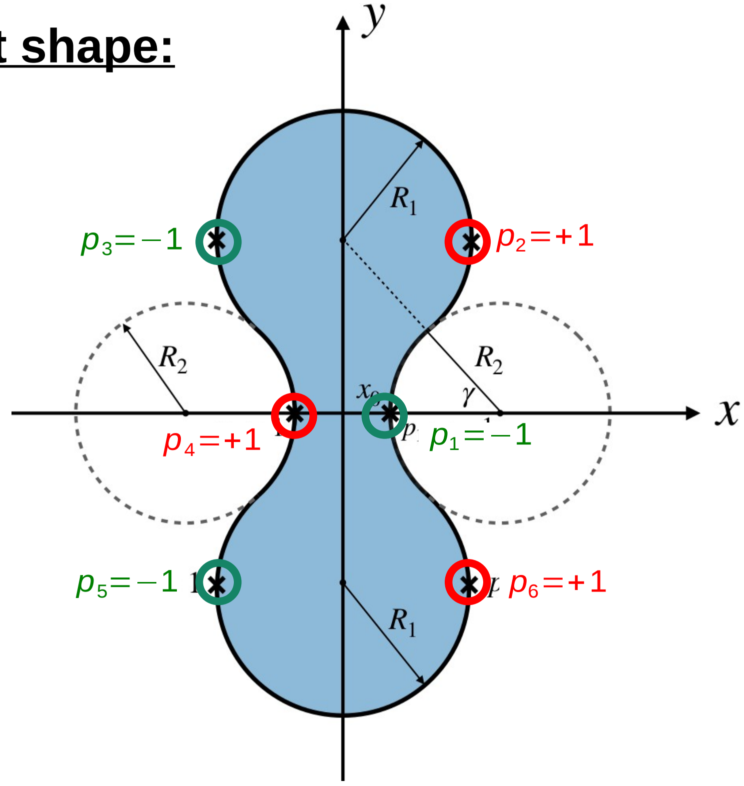
Partner potentials:

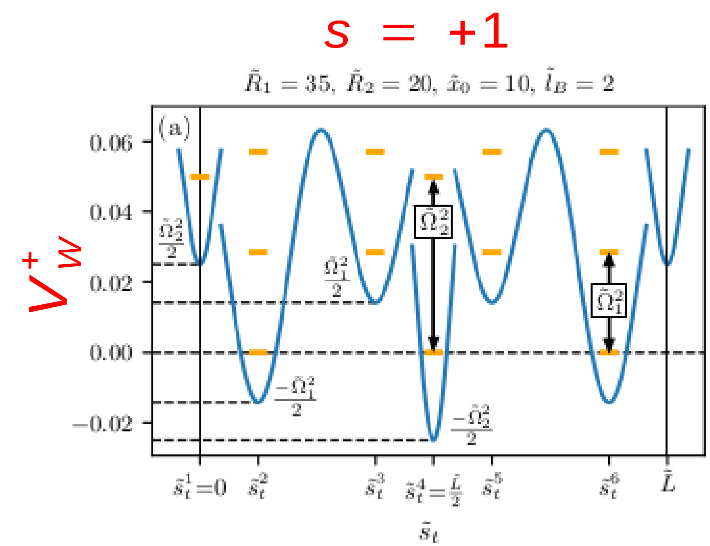
$$V_W^{\pm}(s_t) = (E_Z)^2_{normal}(s_t) \mp \alpha \, \partial_{s_t} (E_Z)_{normal}(s_t)$$

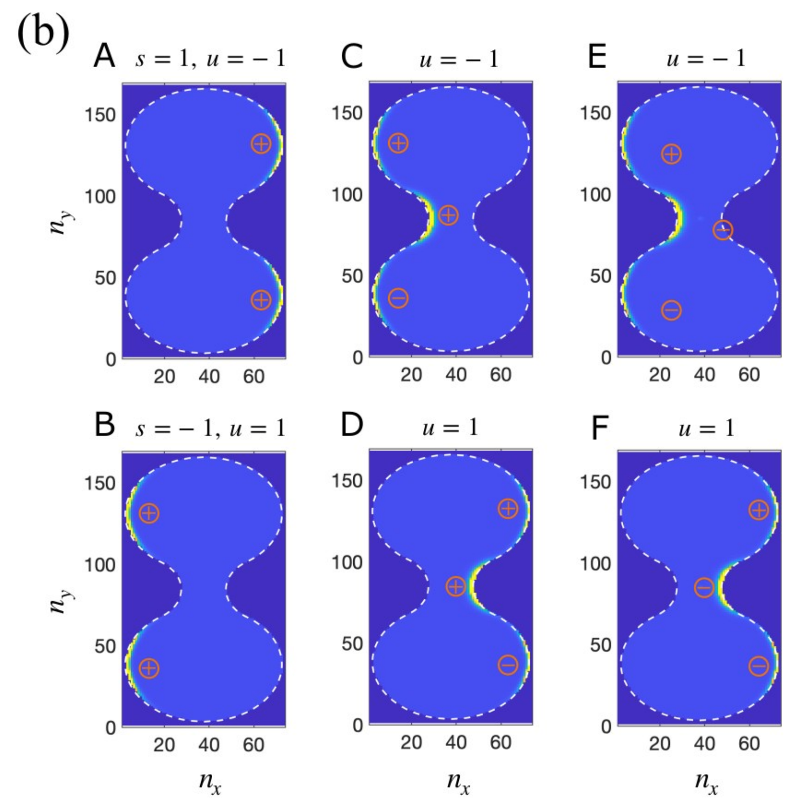
- **Realization of the whole class of periodic Witten models**
- **Mirror symmetric surface => SUSY**
- **Universal low-energy theory for the fundamental relation between 2nd-order topology and SUSY**

Peanut shape:

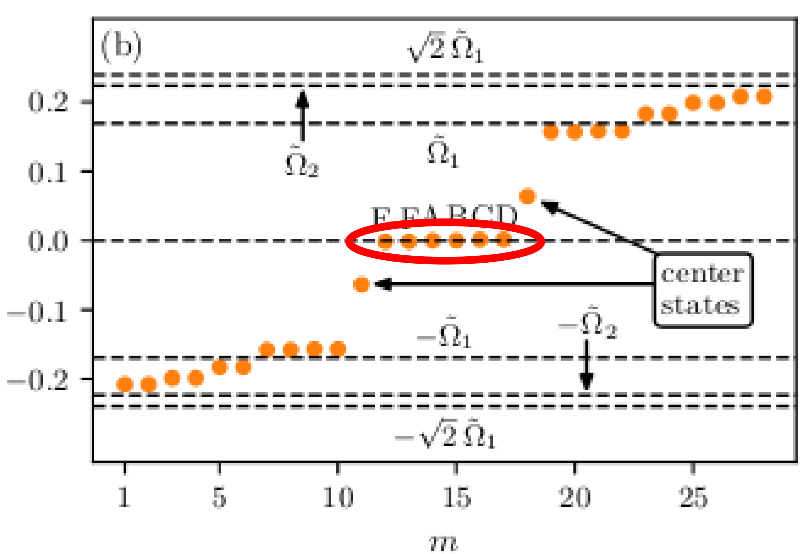




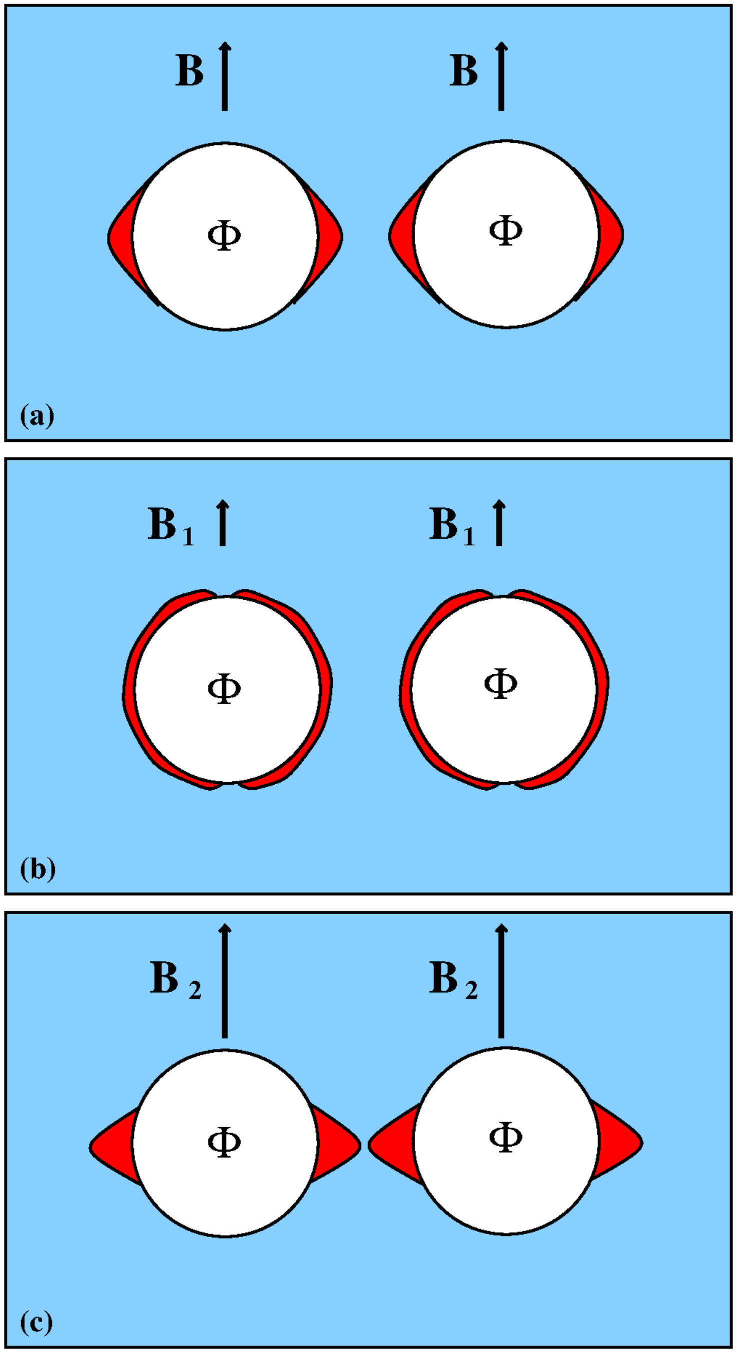




Spectrum:



Topological engineering



use hole states in TP regime: $\tilde{E}_Z < \tilde{\delta} < 1+\tilde{E}_Z$

$$\tilde{\xi}_n = \left(1-\sqrt{1-\tilde{\delta}+\tilde{E}_Z}\right)^{-1}$$

$$\tilde{\xi}_t = \sqrt{\tilde{R}\,\tilde{l}_B} \ll \tilde{R}$$

increase B

$\Rightarrow$ ξ_n increases
ξ_t decreases

$$\Delta_{bulk} = |\delta| - E_Z$$

$$\Delta_{surface} \approx E_Z$$

decrease $B_1 < B$

$\Rightarrow$ increase coupling between two states at the same hole by detuning flux from 1/2

increase $B_2 > B$

$\Rightarrow$ increase coupling between states at different holes

Summary

(1) band inversion + spin-orbit + Zeeman: → minimal model for 2nd-order TI in 2D
→ very flexible topological states

(2) half-integer AB-flux: → SUSY spectrum: anticommuting inversion + mirror symmetry
→ topological states protected by SUSY + chiral symmetry

(3) smooth surfaces: → derivation of effective surface Hamiltonian
→ generic class of periodic Witten models
→ universal low-energy theory: 2nd-order topology ↔ SUSY
→ trapping of topological states in effective surface potentials

(4) Generalization to 3D: → anomalous 3D-QHE via hinge states on torus geometry
→ Z. Hou et al., Phys. Rev. B 107, 075437 (2023)