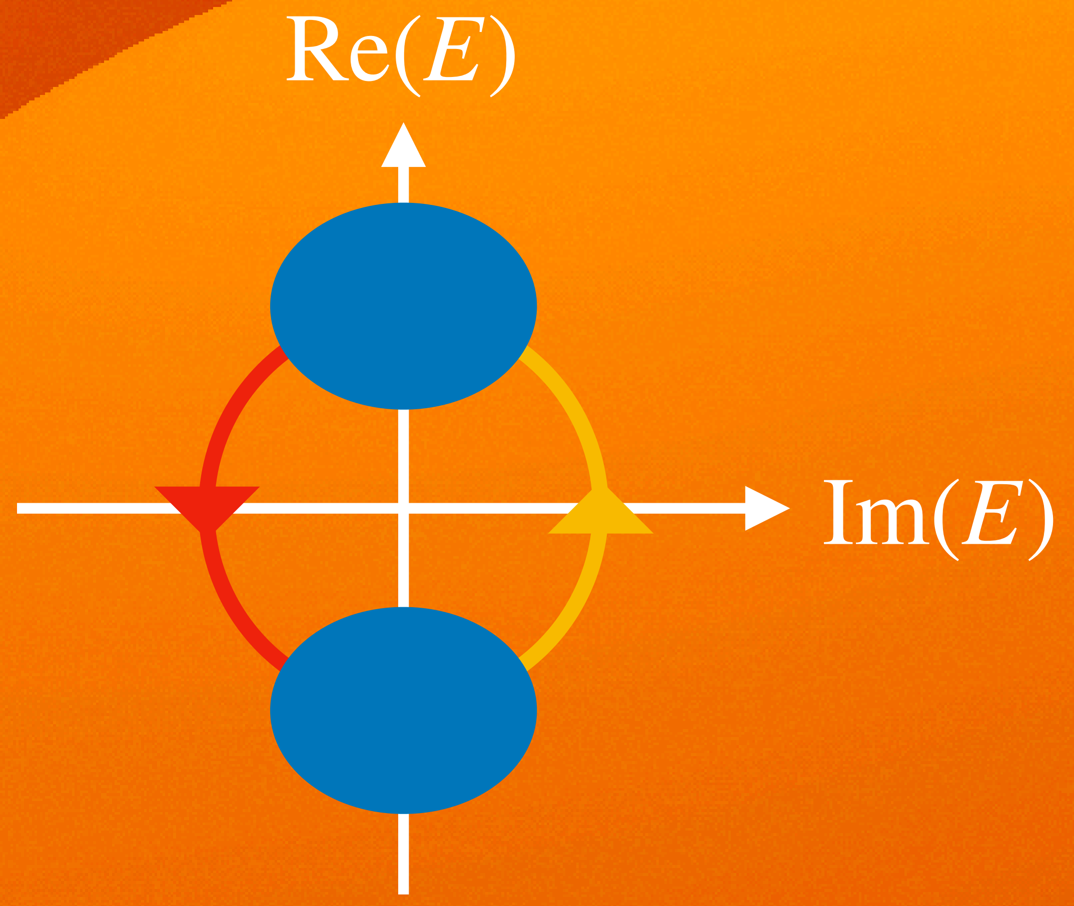
Hermitian Bulk — Non-Hermitian Boundary Correspondence

Frank Schindler,
Princeton U & Imperial College



arXiv:2304.03742
with Kaiyuan Gu, Biao Lian, and Kohei Kawabata

Introduction: Electronic band structure

Context: **electronic structure** of crystalline materials

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Non-interacting electrons —> focus on **1 electron**

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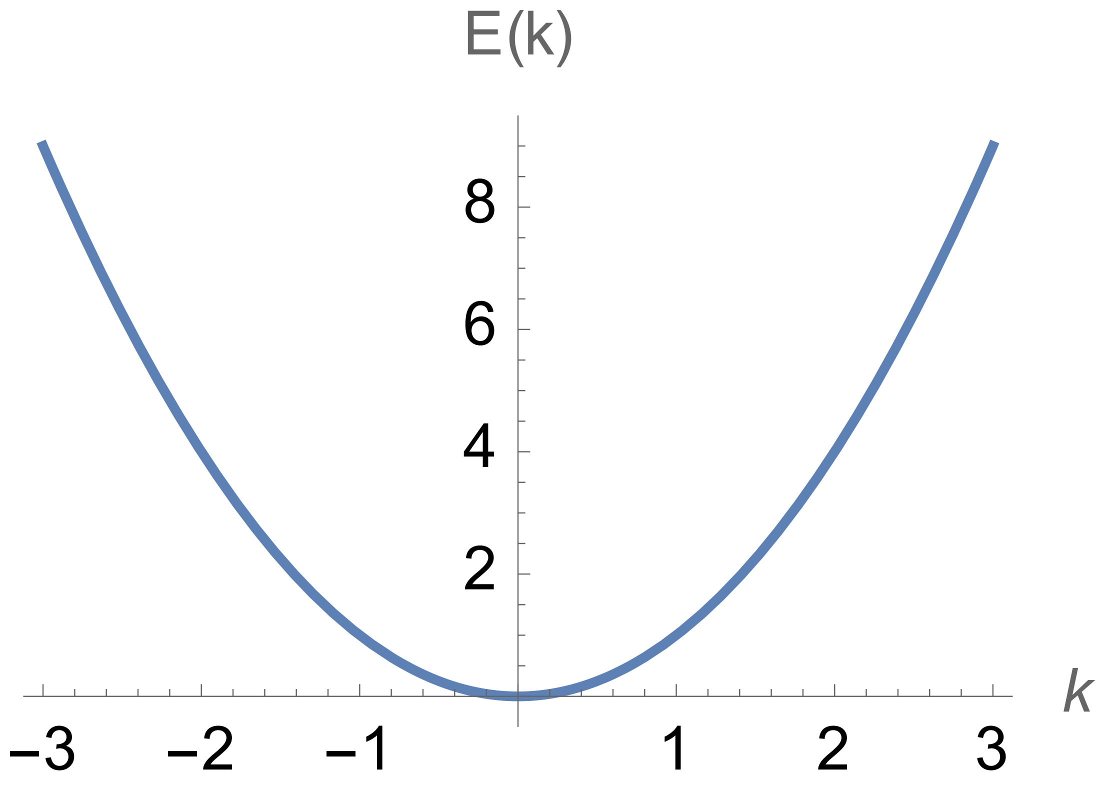
Hermitian Hamiltonian: **real** energies



1 electron in **free space**: $H(k) = \dfrac{k^2}{2m}$

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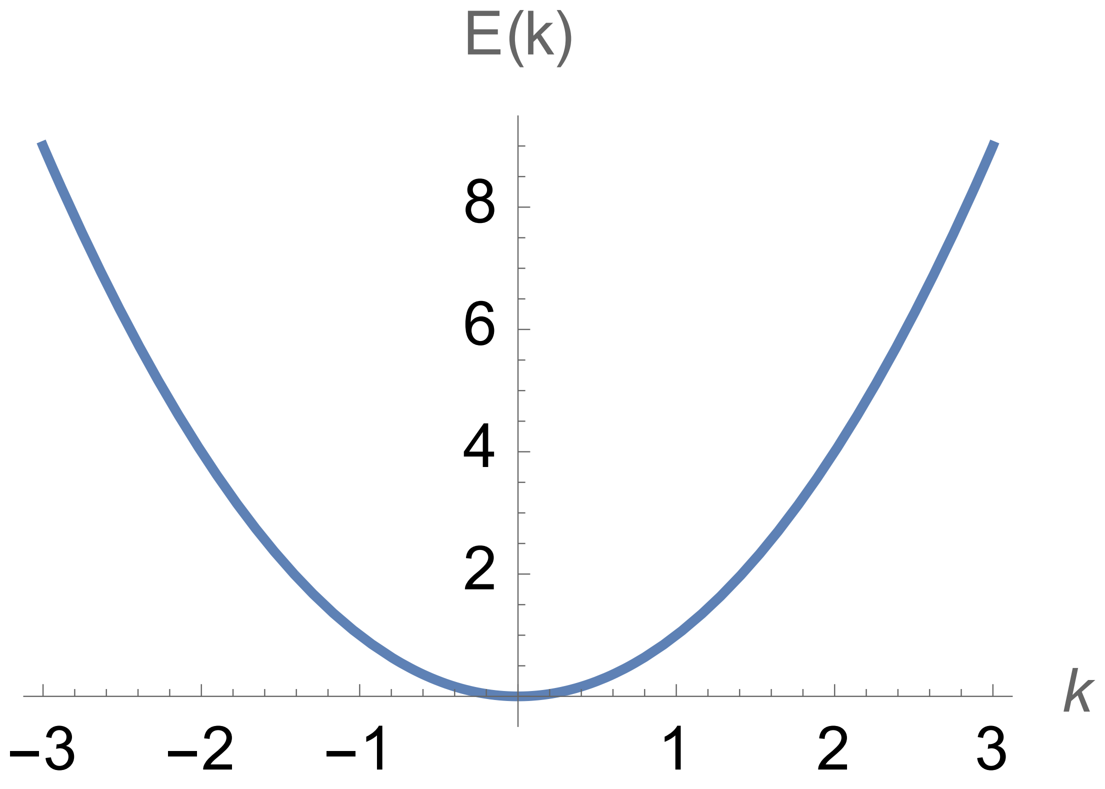

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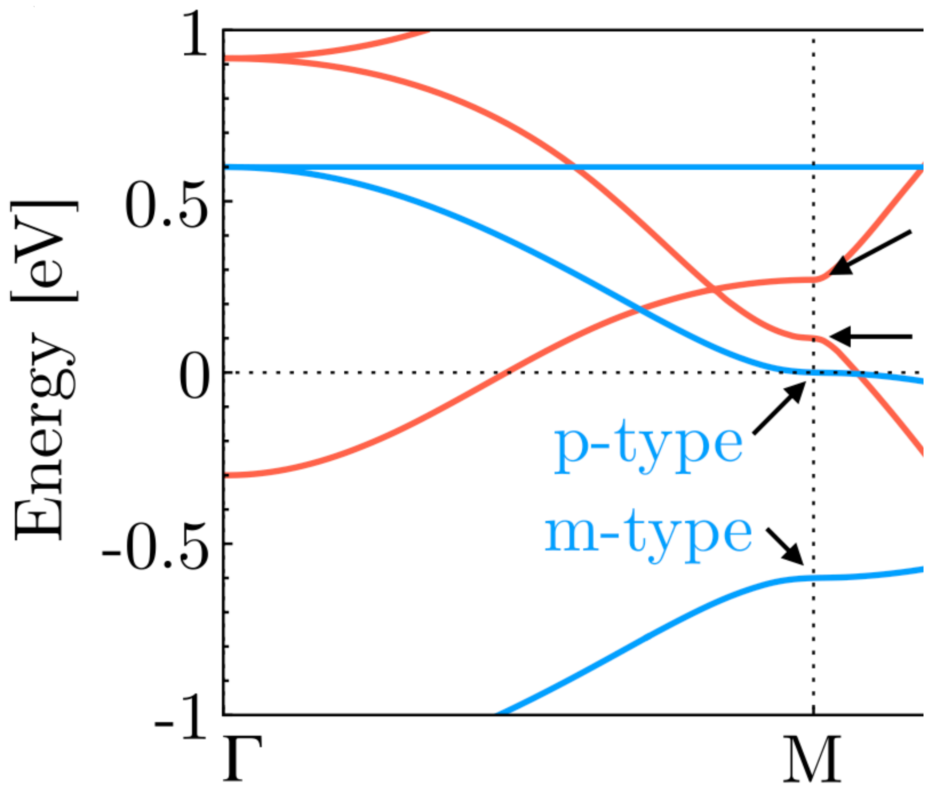


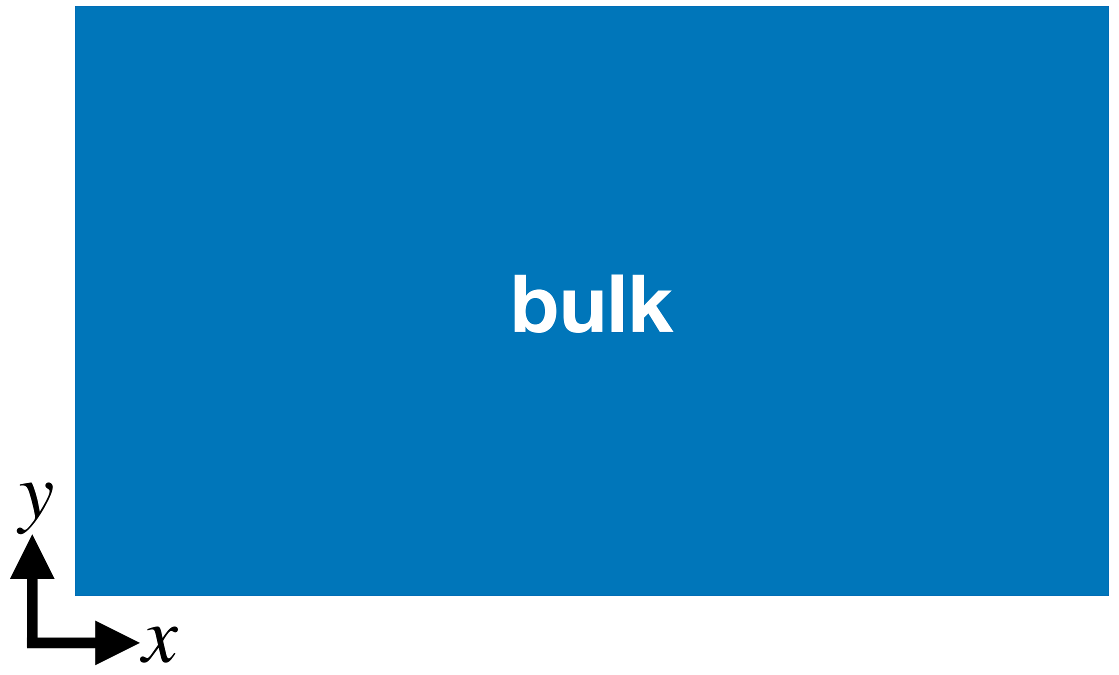
eigenvalues are **energy bands**

1 electron in a **crystal**: $H(k)$ is a **matrix**!

Hermitian Bulk-Boundary Correspondence

Simplest example: 2D **Chern insulator** (=quantum Hall effect)

real space geometry



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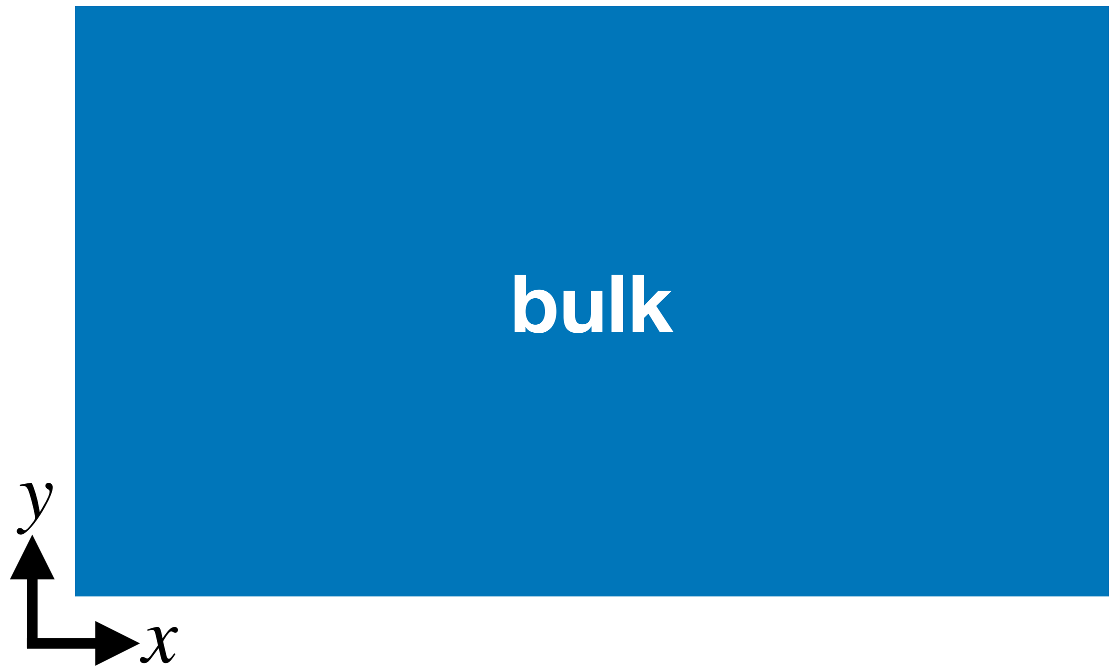

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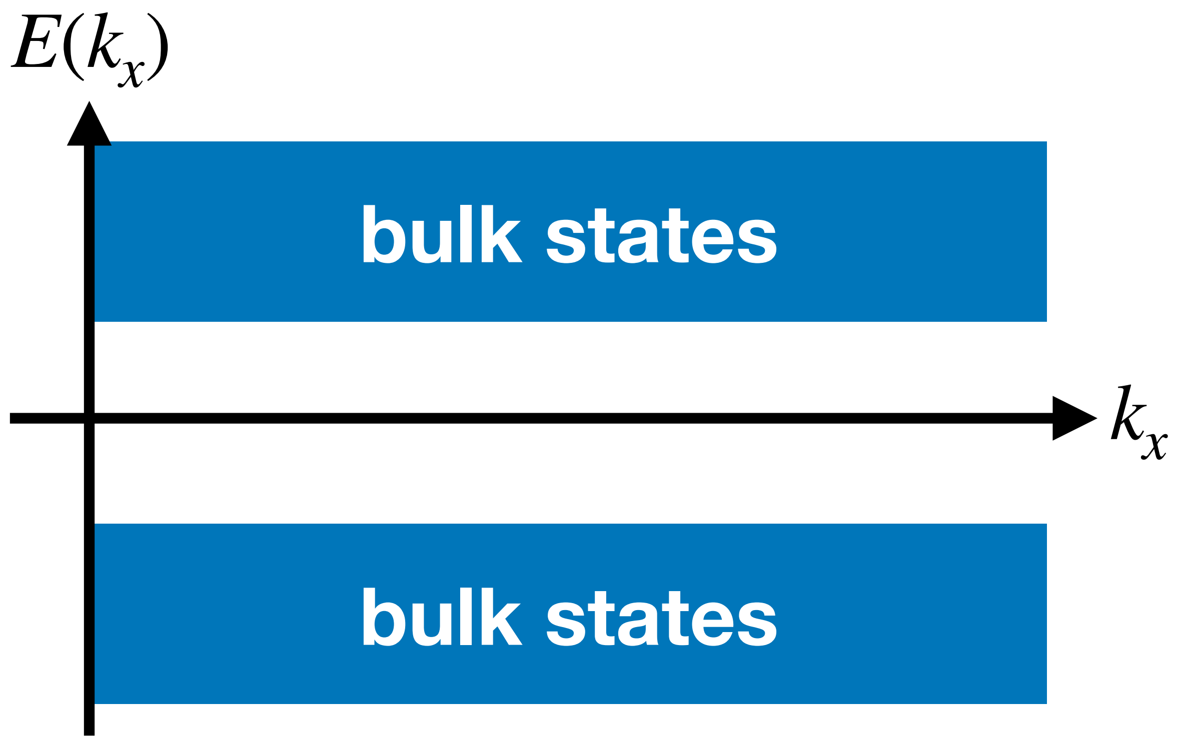

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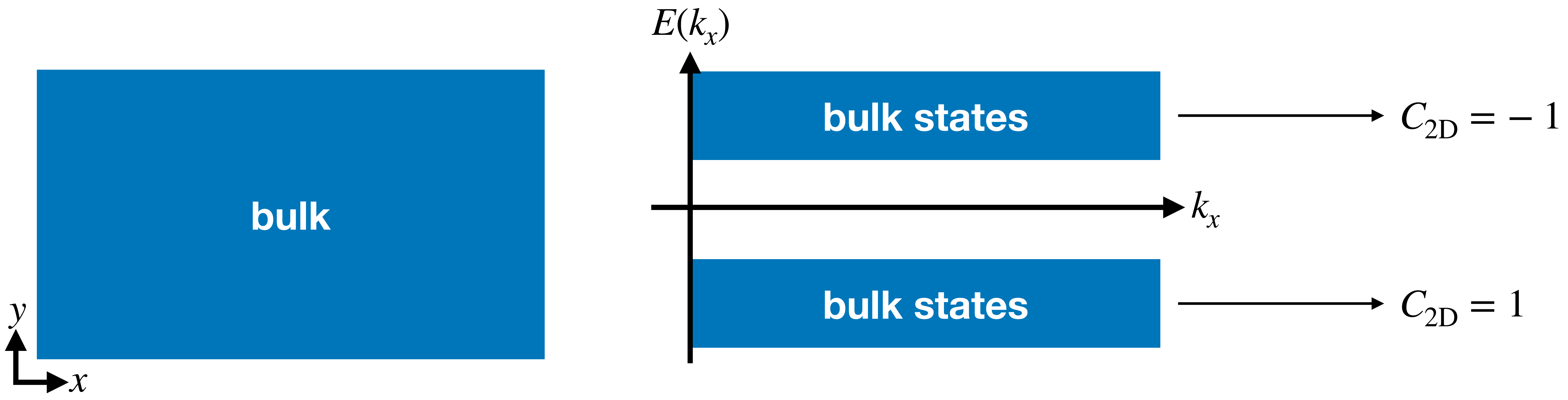

real space geometry

energy bands

topological invariant:
Chern number $C_{2D} \in \mathbb{Z}$

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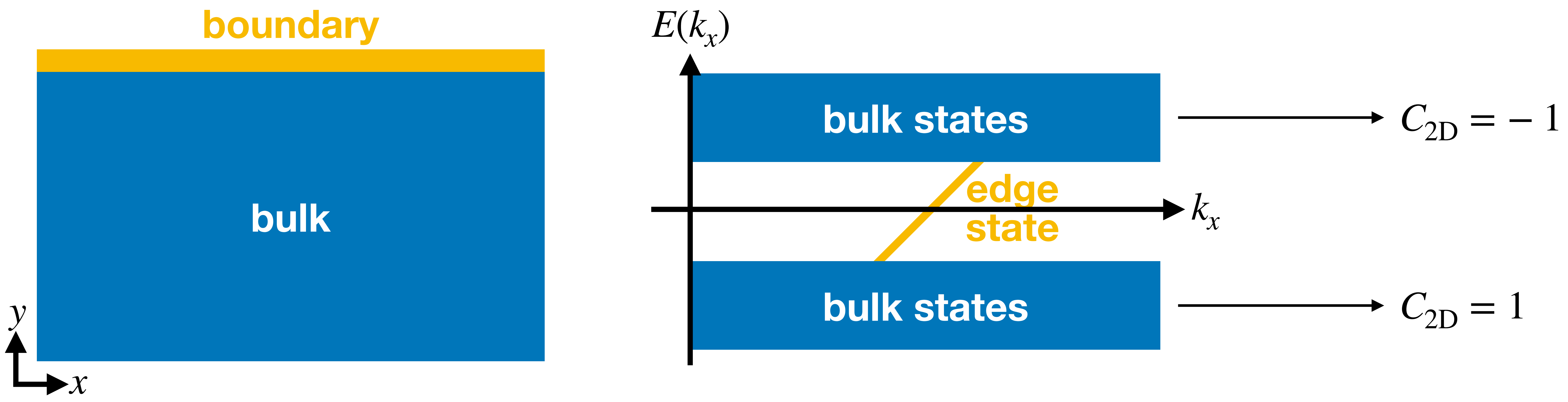
real space geometry | **energy bands** | topological invariant: **Chern number** $C_{2D} \in \mathbb{Z}$



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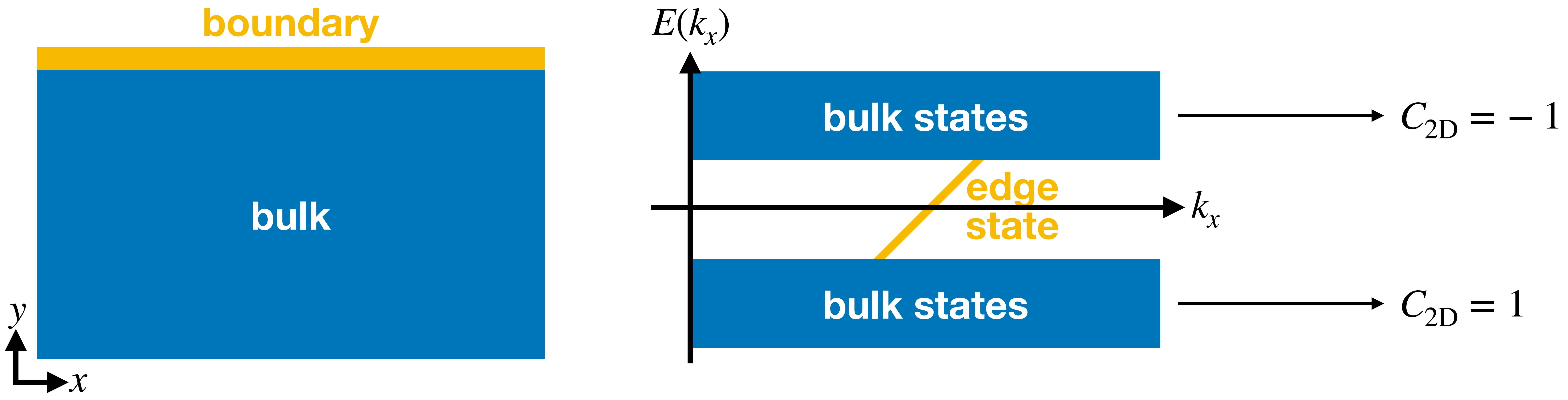

boundary

bulk

y

x

$E(k_x)$

bulk states

edge state

k_x

bulk states

$C_{2\mathrm{D}} = -1$

$C_{2\mathrm{D}} = 1$

Bulk-boundary correspondence:

$$C_{2\mathrm{D}} = N_{\mathrm{right-movers}} - N_{\mathrm{left-movers}} \equiv \Delta_{1\mathrm{D}}$$

bulk invariant

boundary invariant

Non-Hermitian tight-binding systems

Simplest example: 1D **Hatano-Nelson chain** (=non-Hermitian skin effect)

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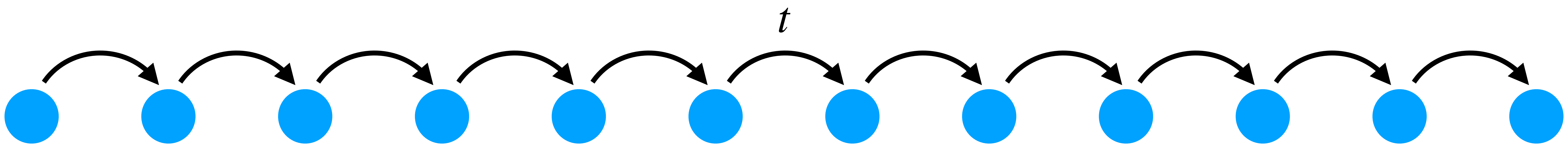

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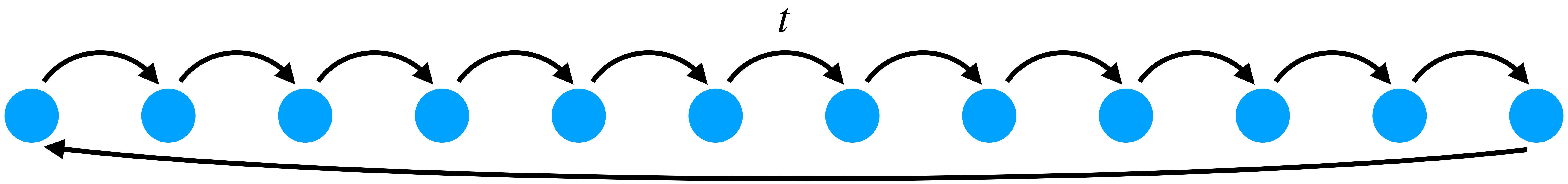
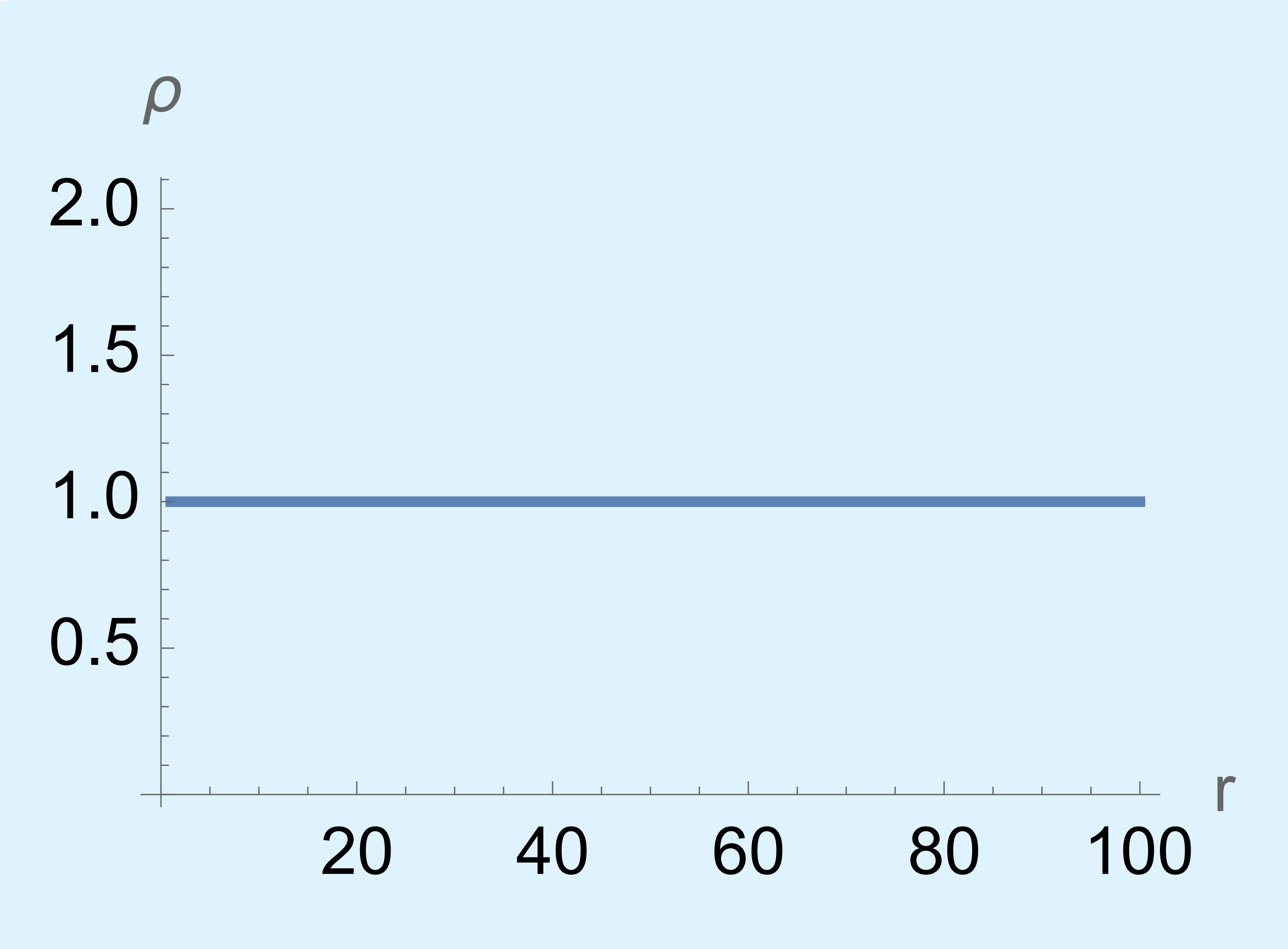


periodic boundary conditions



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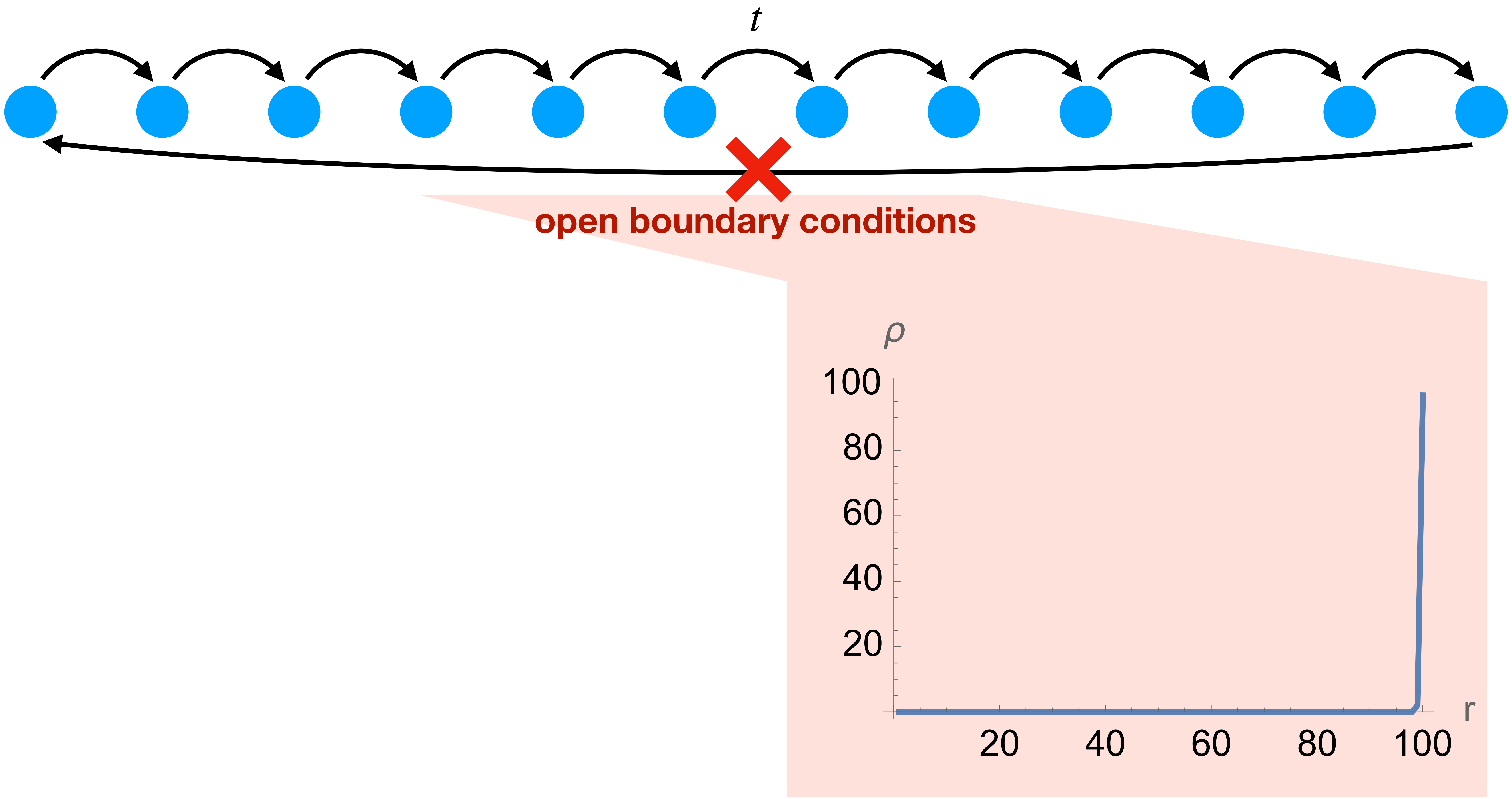

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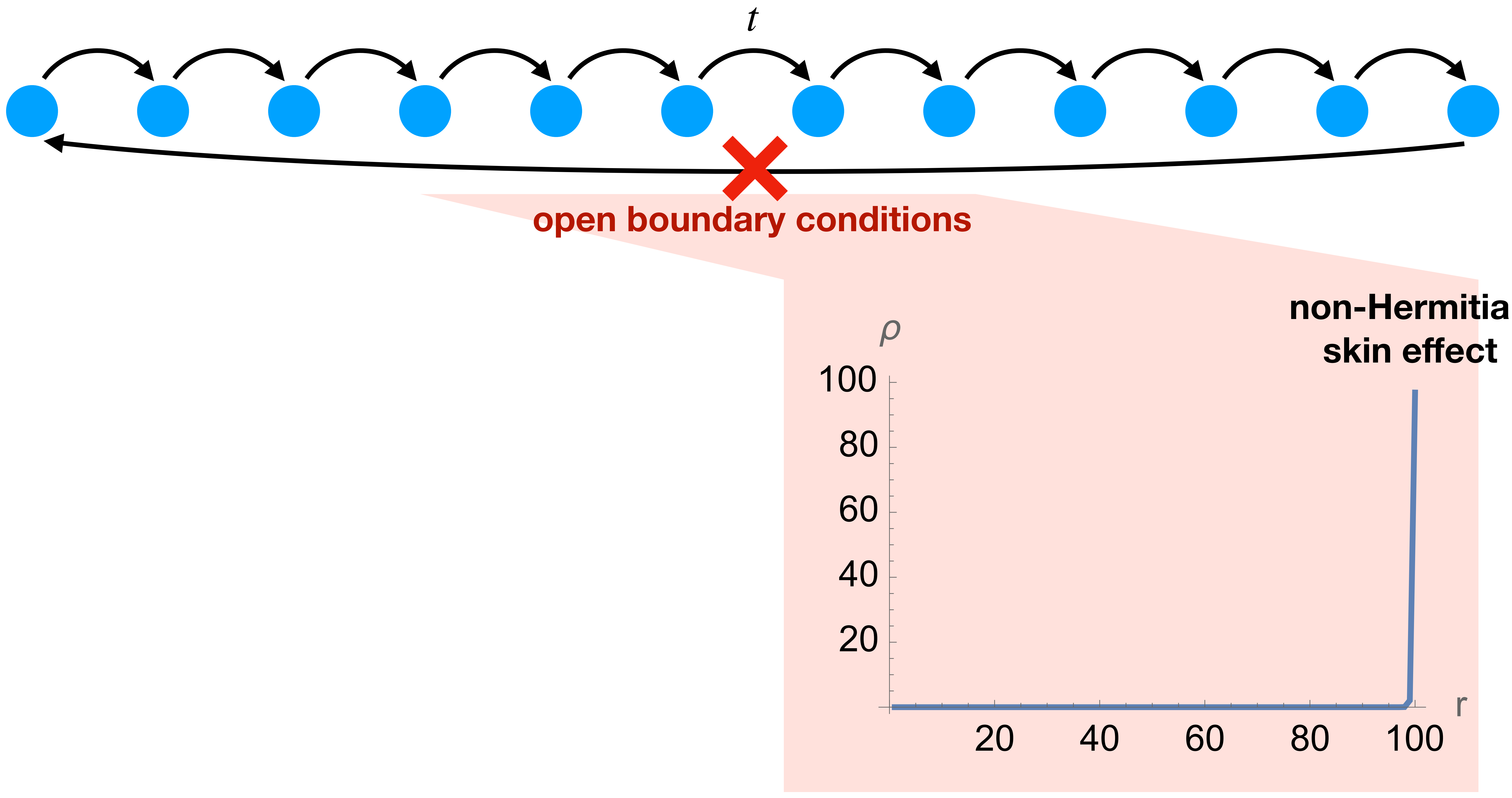
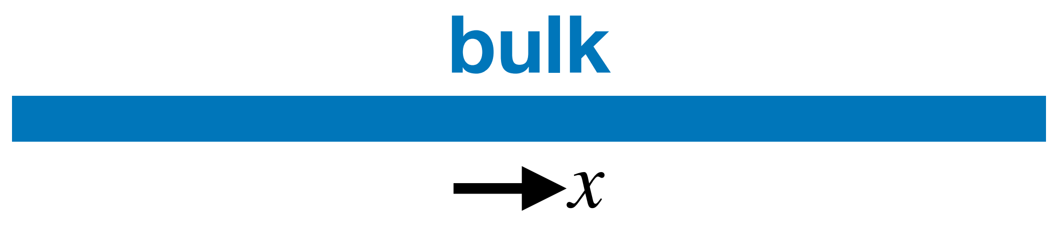
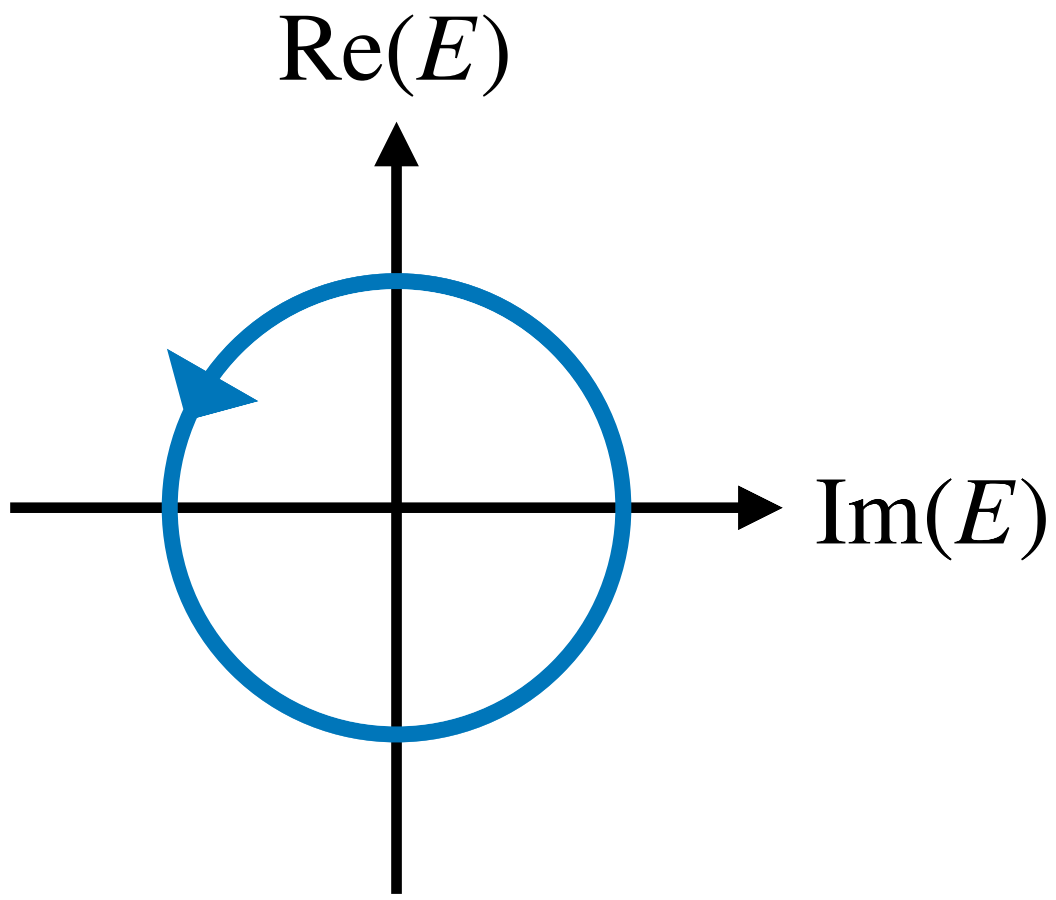


Non-Hermitian Bulk-Boundary Correspondence

Hatano-Nelson **Non-Hermitian Hamiltonian**: $H(k) = te^{\mathrm{i}k}$





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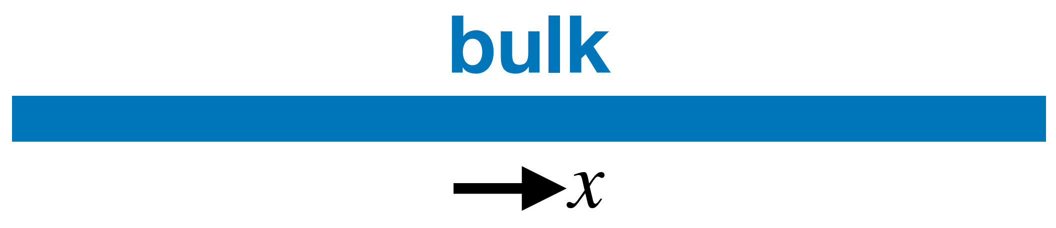

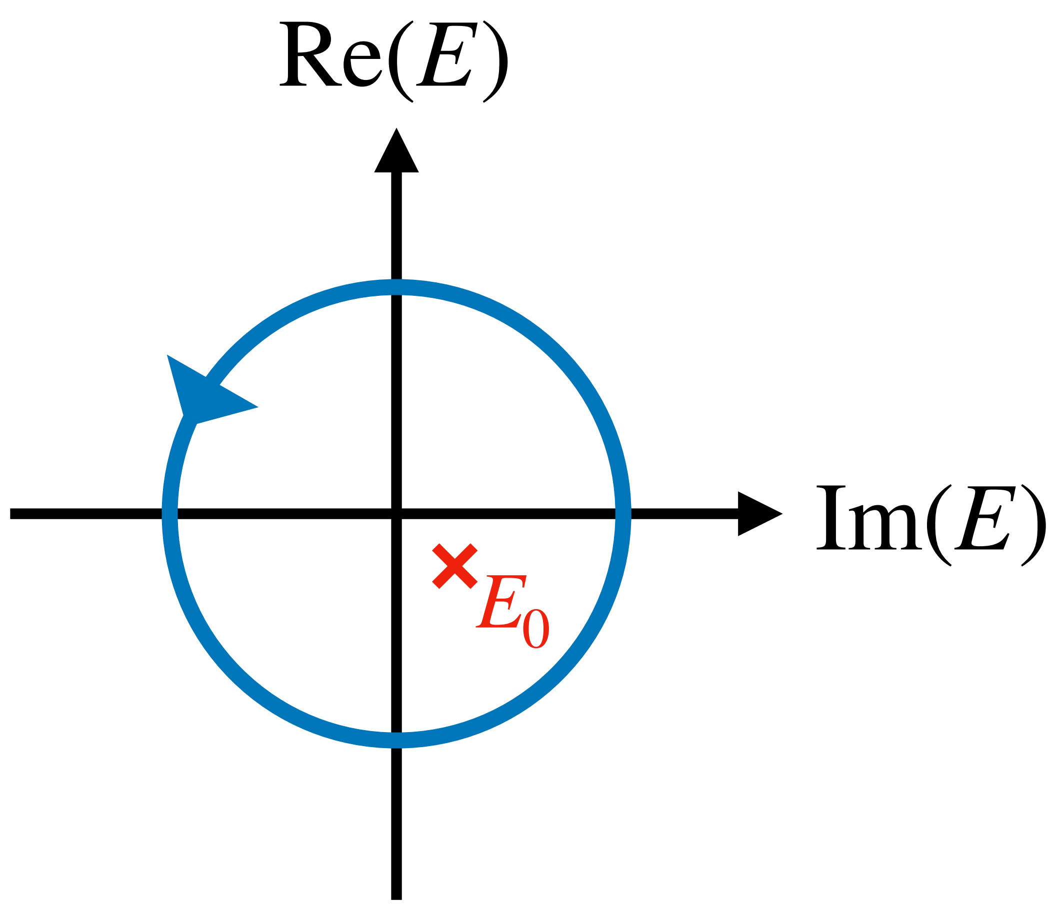

winding number invariant:

$$W_{1\mathrm{D}}(E_0) = \int \frac{\mathrm{d}k}{2\pi\mathrm{i}} \frac{\partial}{\partial k} \log\det[H(k) - E_0] = 1$$ for all E_0 inside the **point gap**

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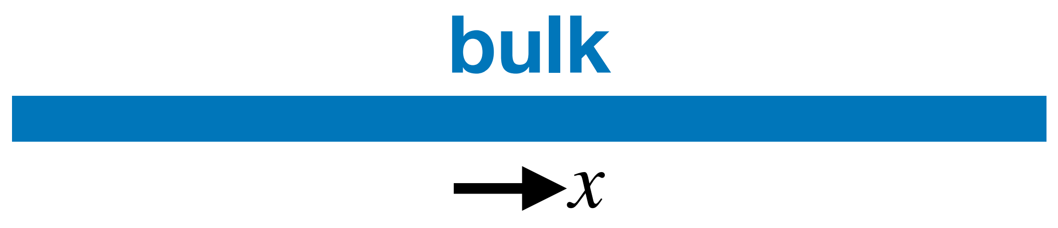

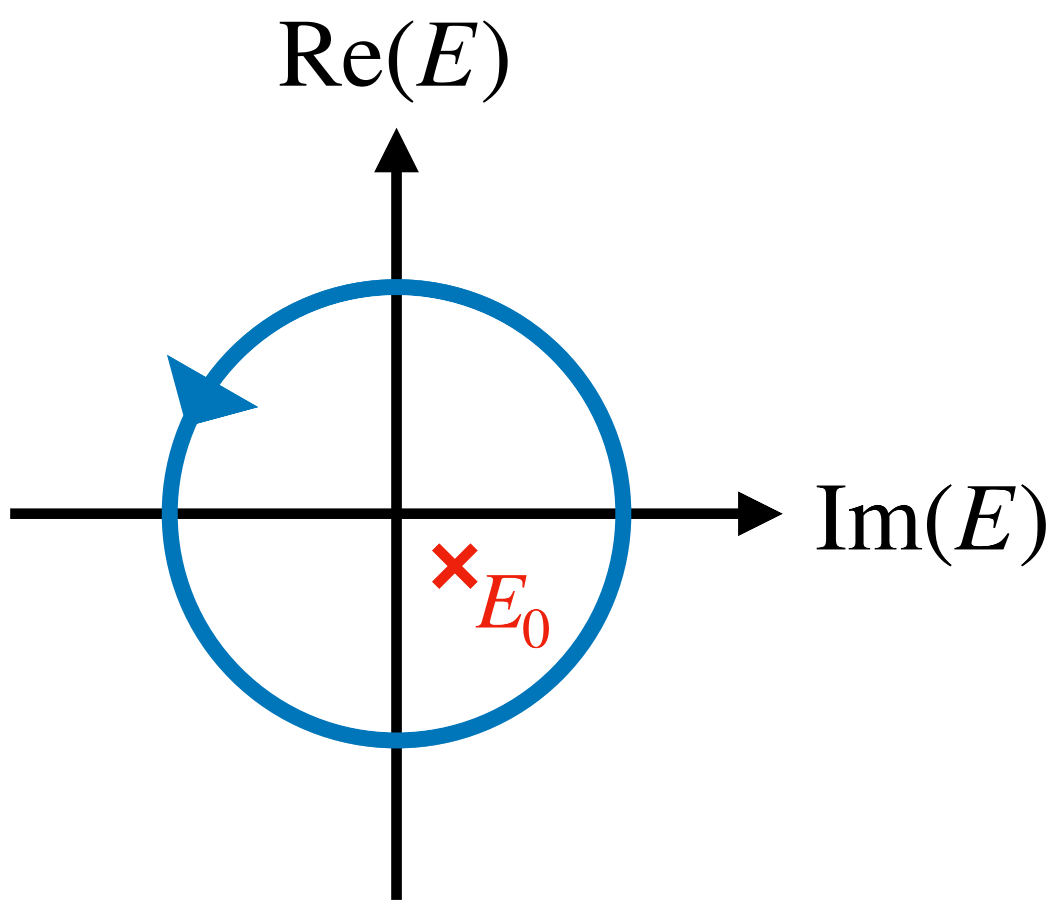

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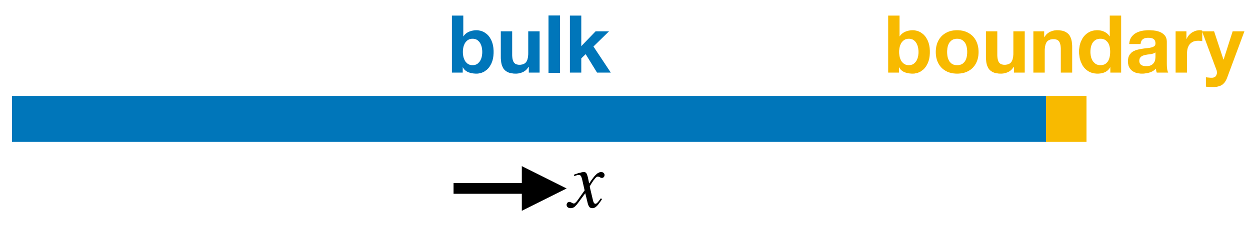

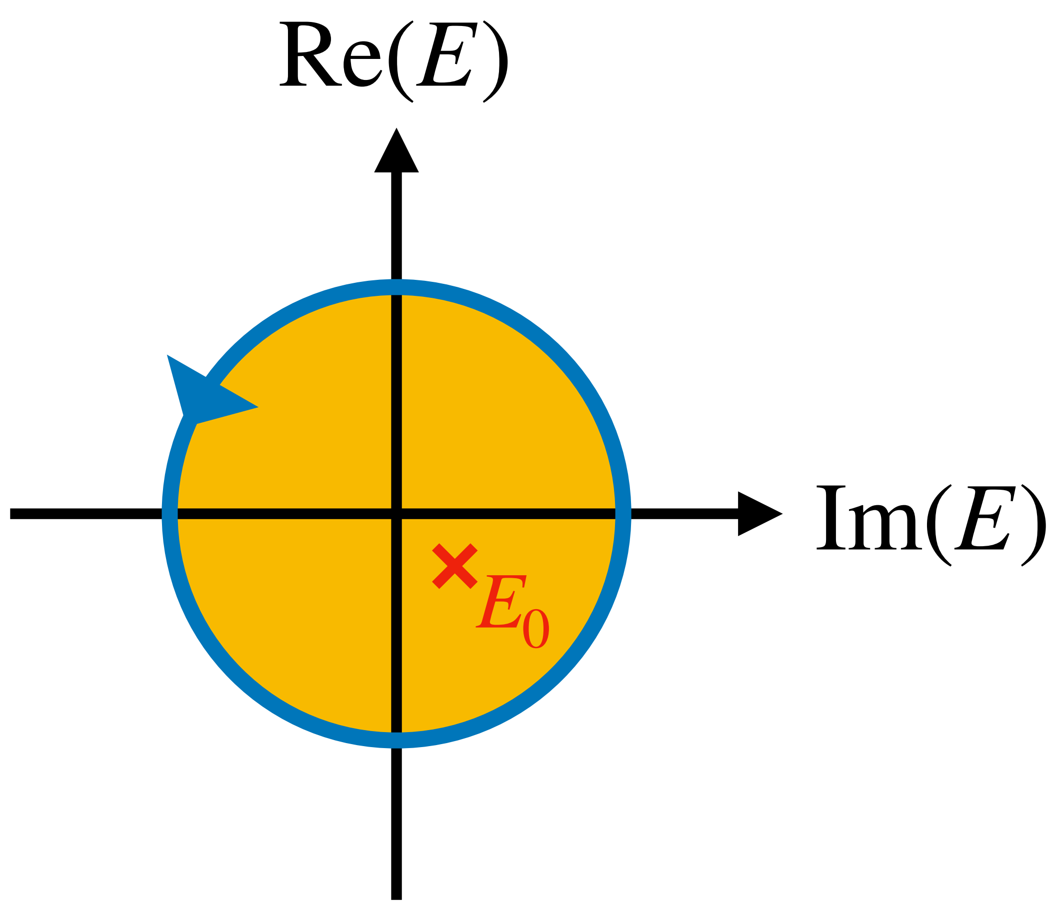

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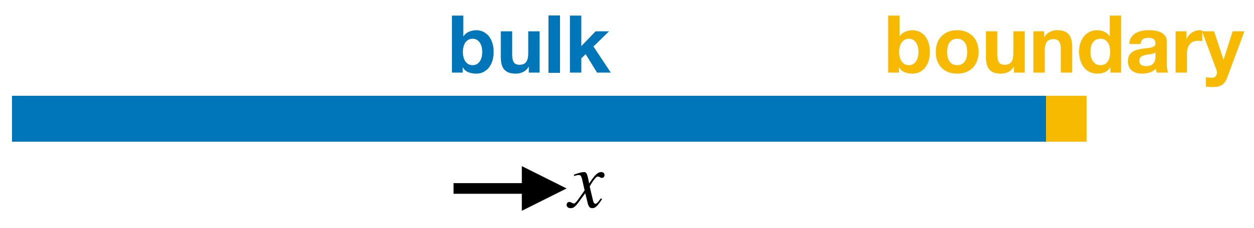

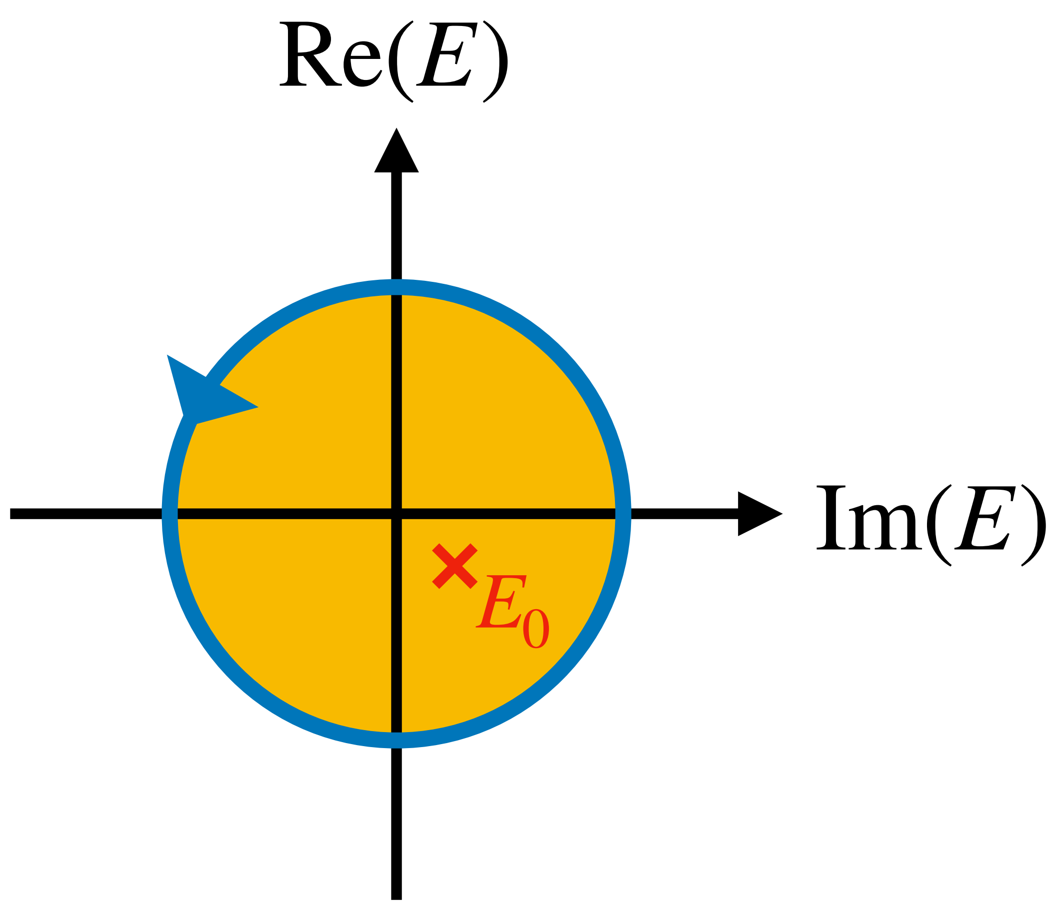

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bulk invariant

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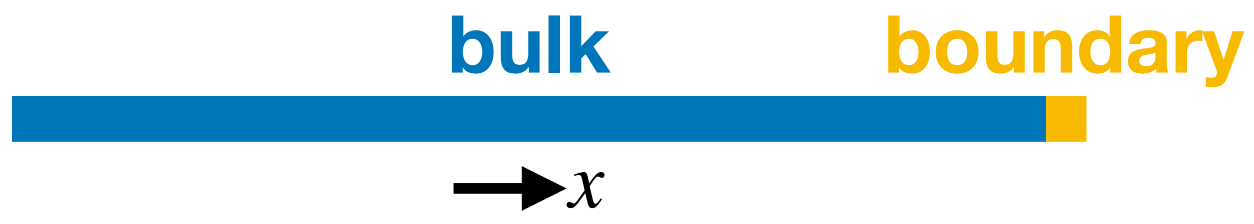

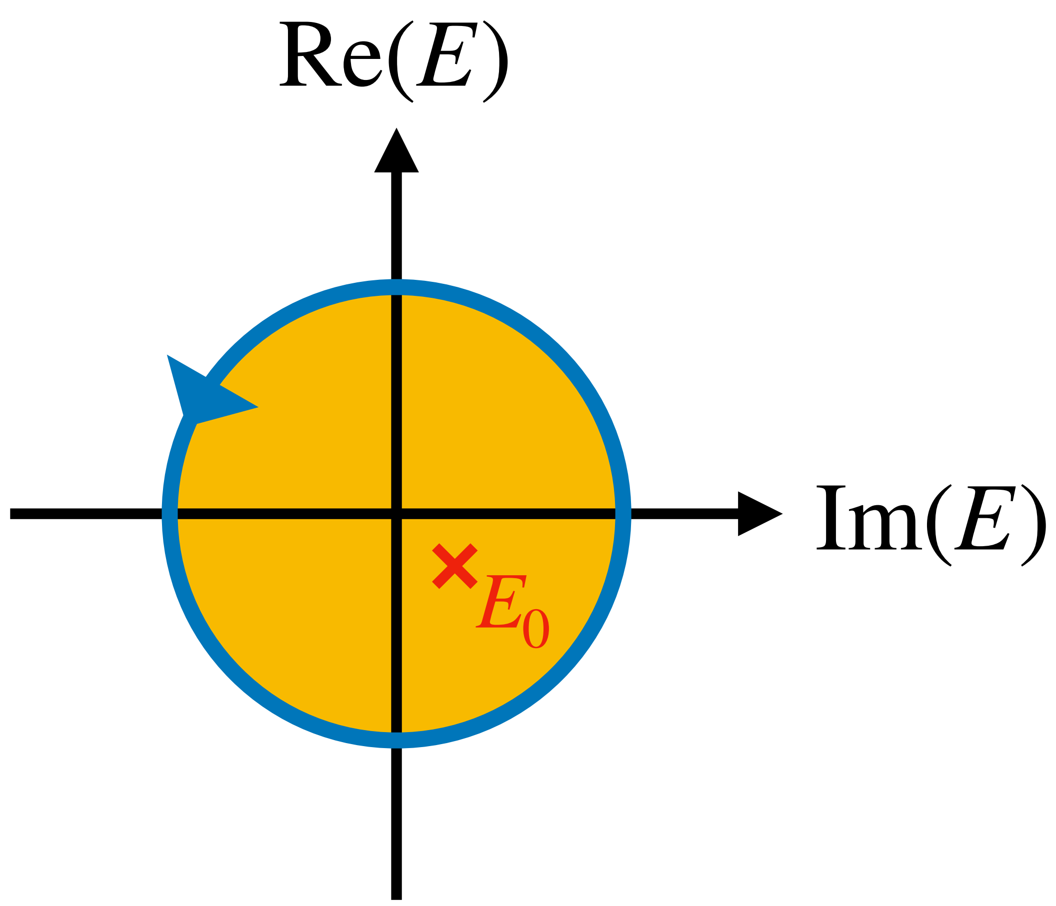

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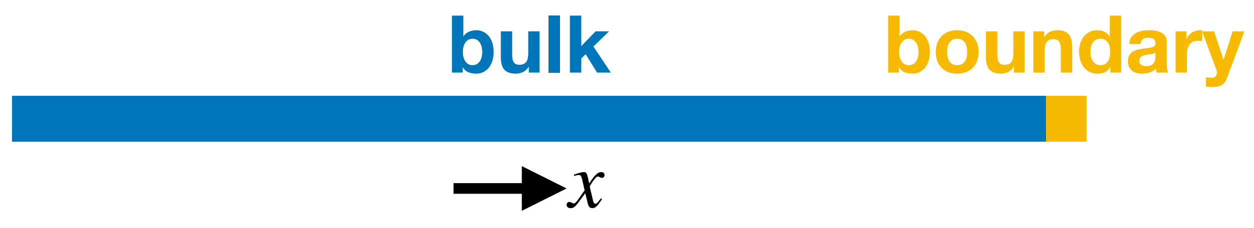

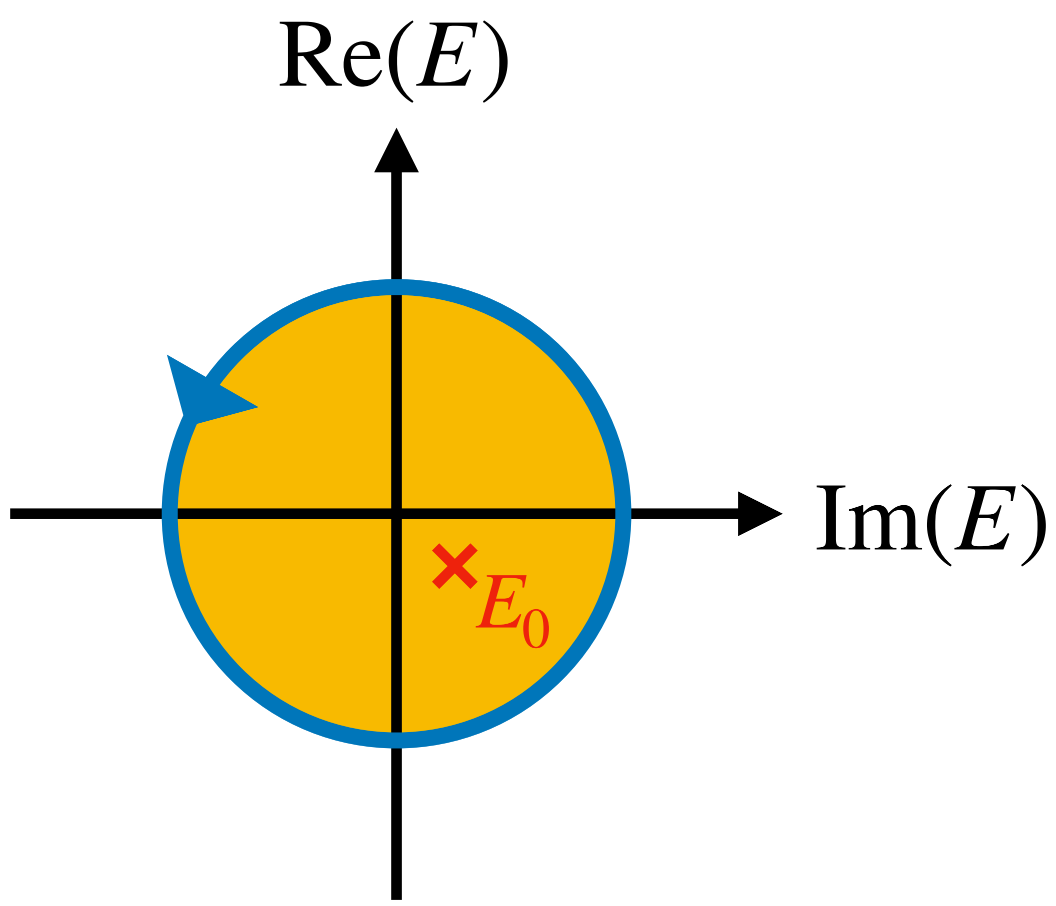

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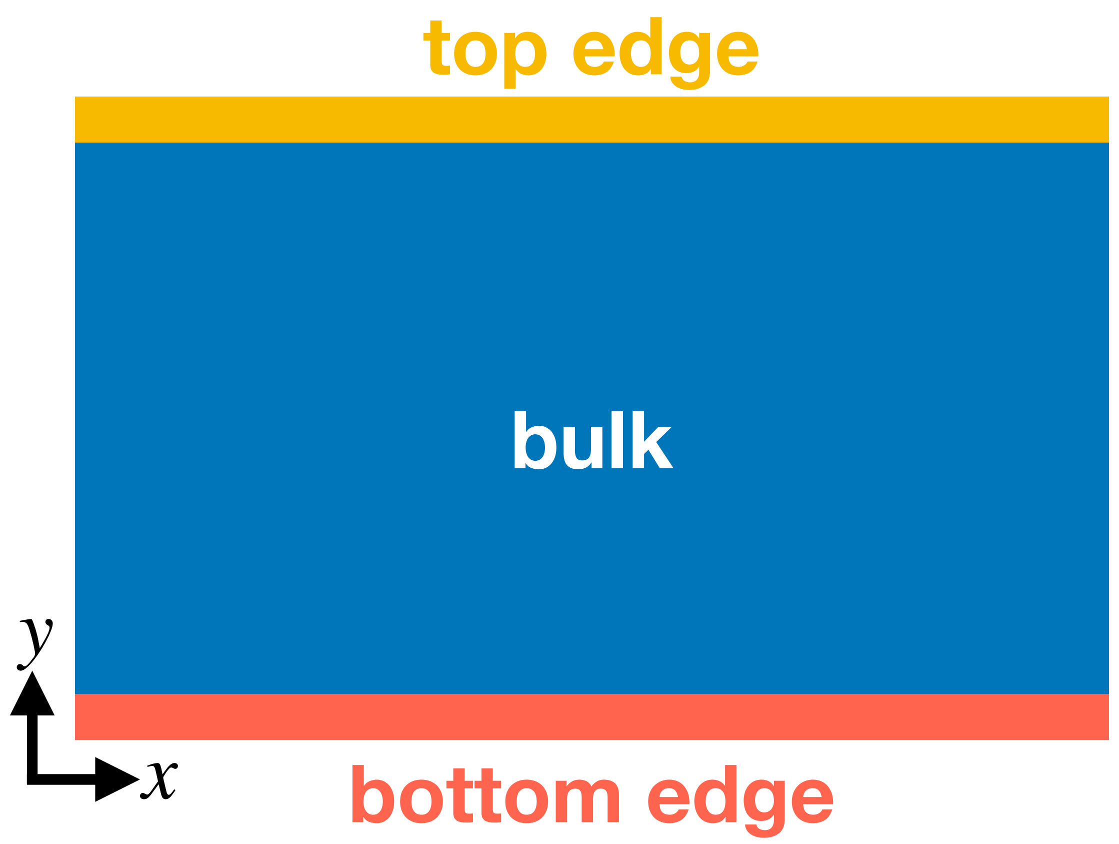
boundary invariant

$H\,|\,\Psi\rangle = E_0\,|\,\Psi\rangle$

$\langle\Psi\,|\,H = \langle\Psi\,|\,E_0$

Hermitian—Non-Hermitian Correspondence

Let’s add **non-Hermitian perturbations** to the **Chern insulator**!



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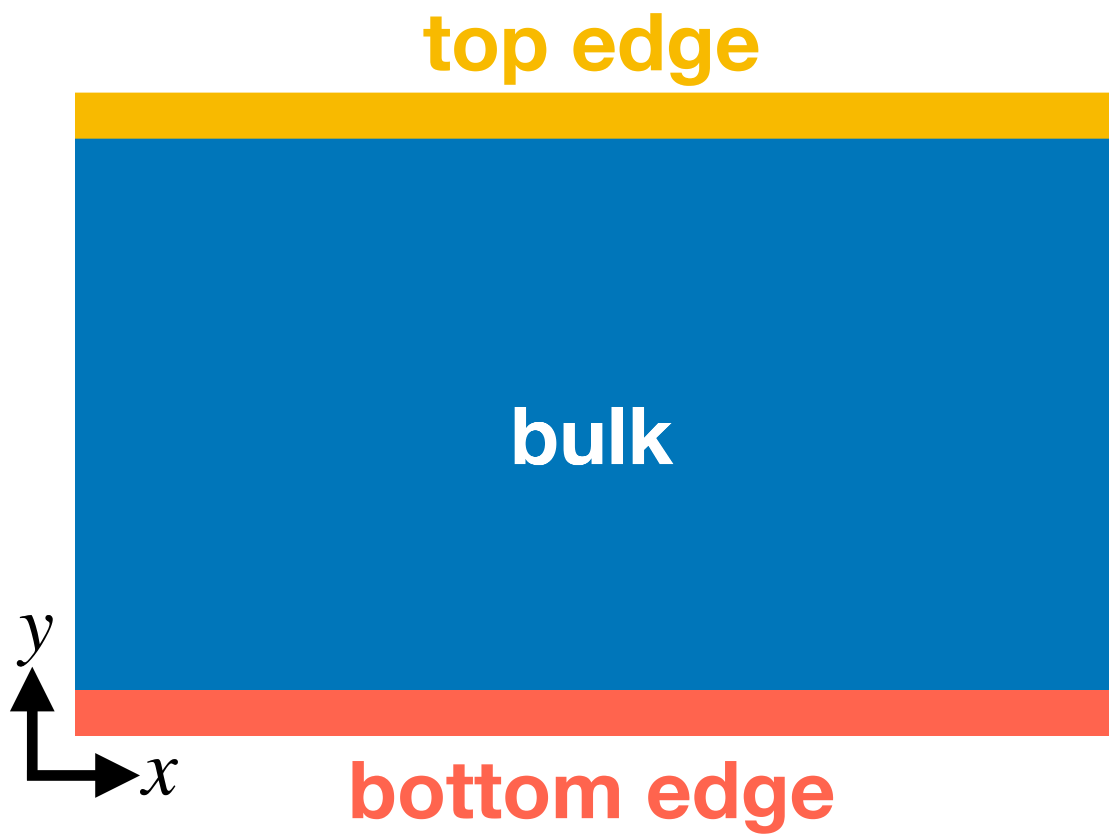

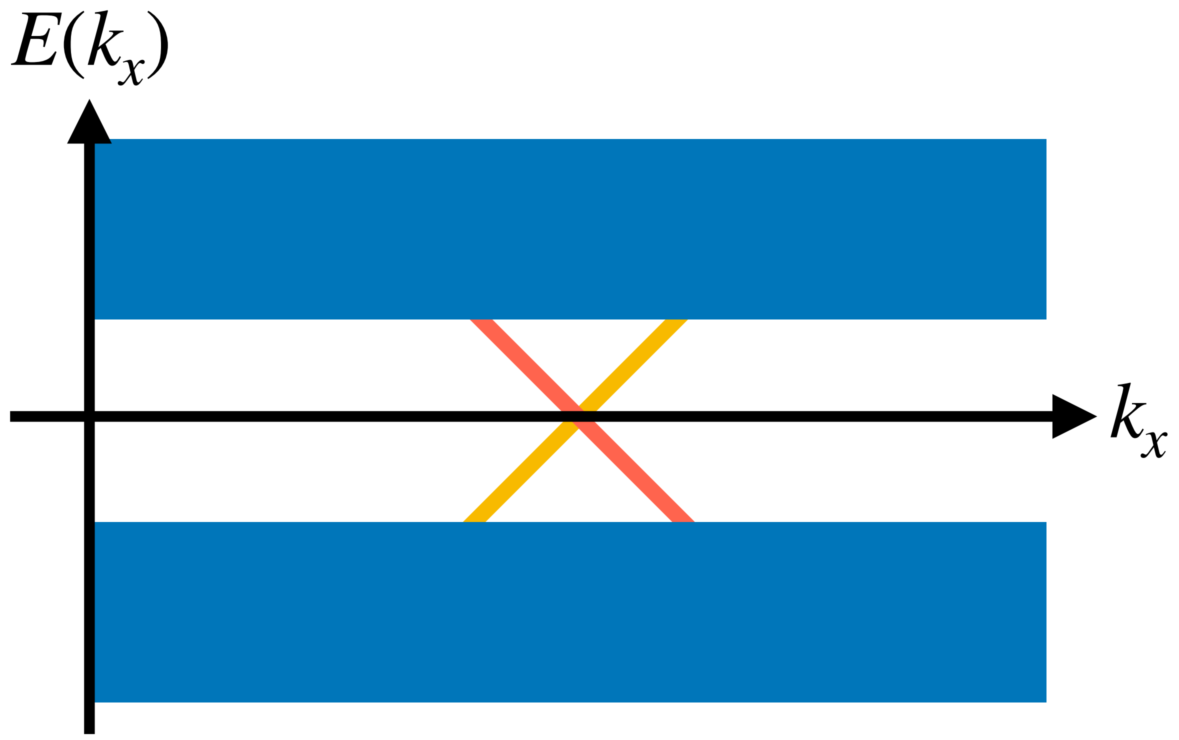

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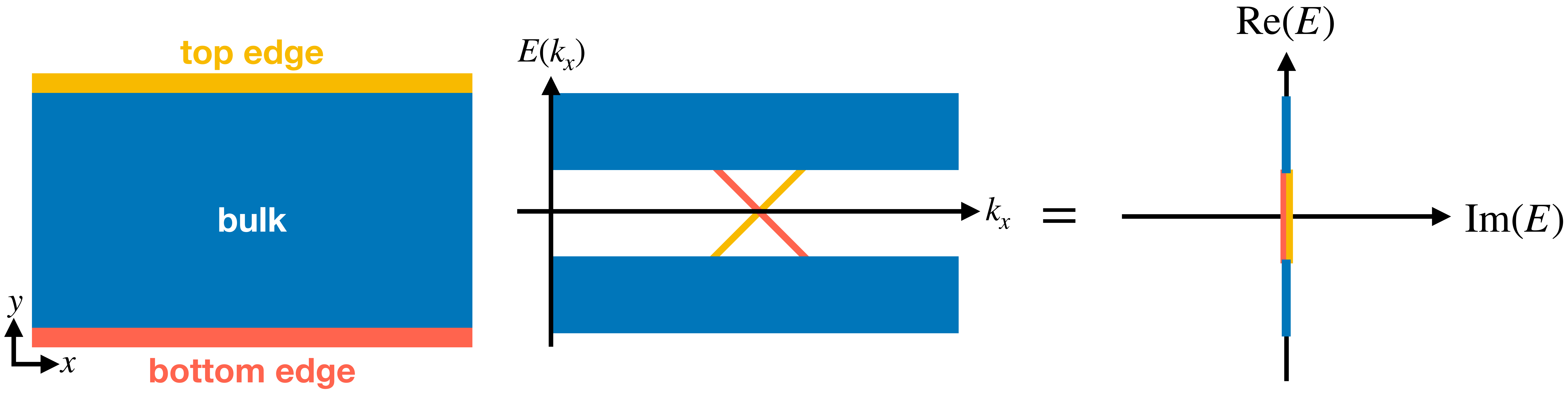

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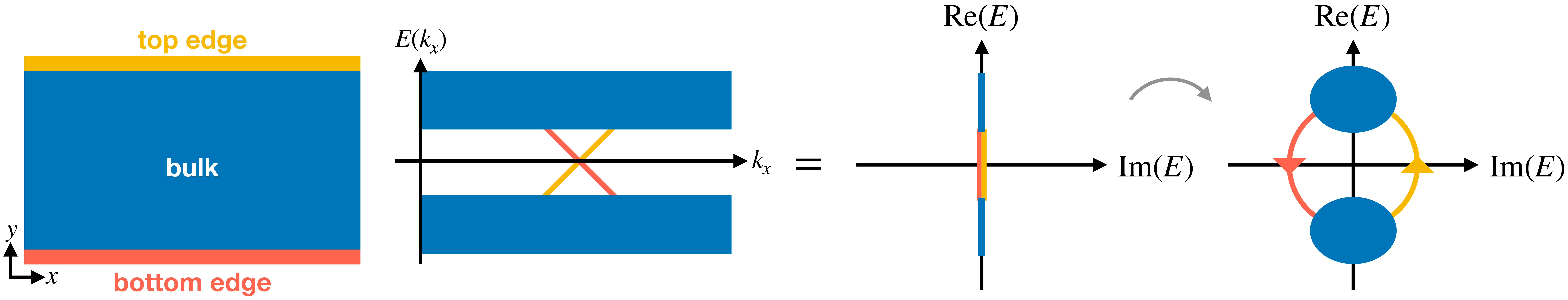

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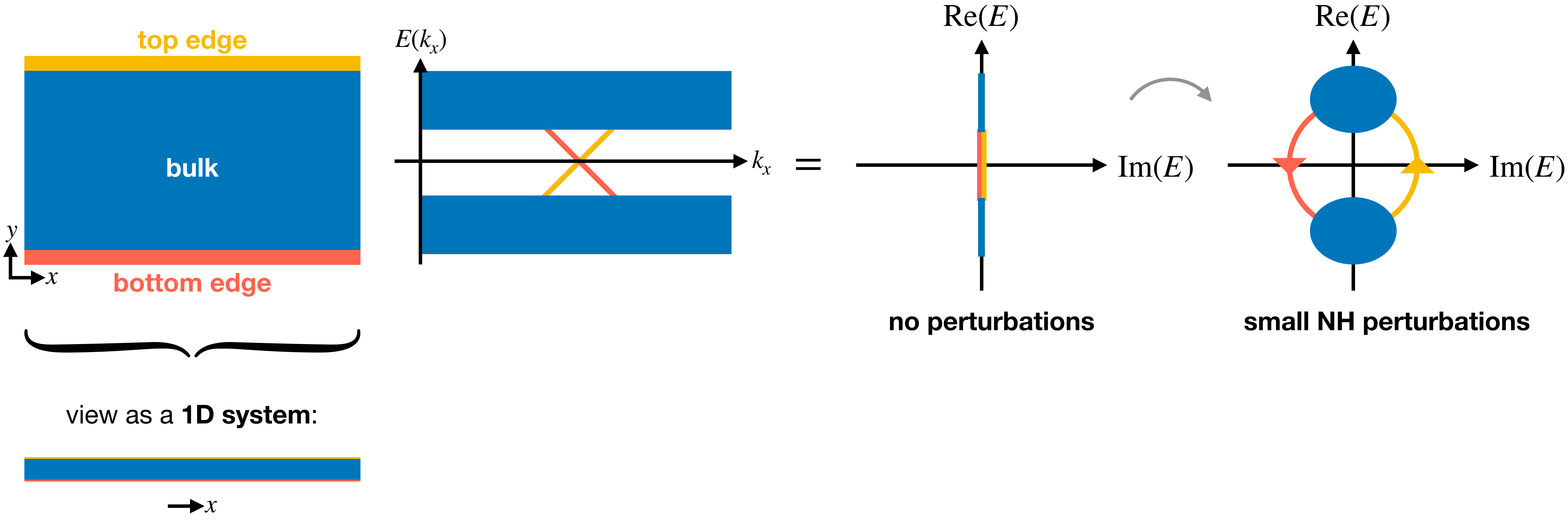

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view as a **1D system**:

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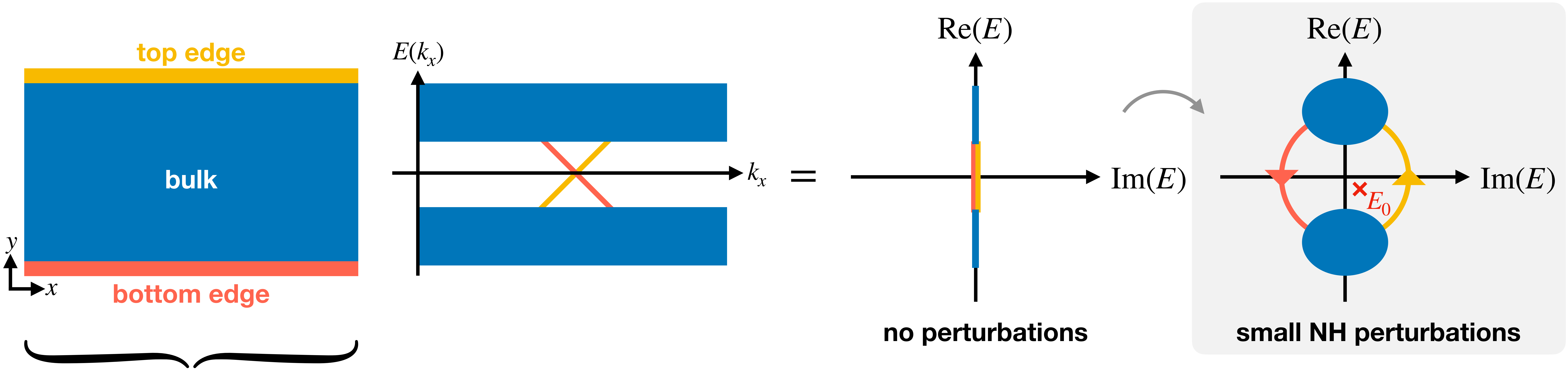

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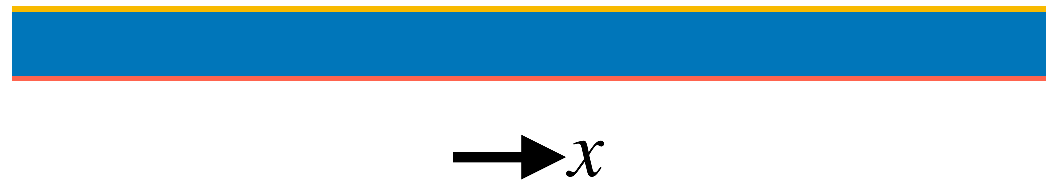

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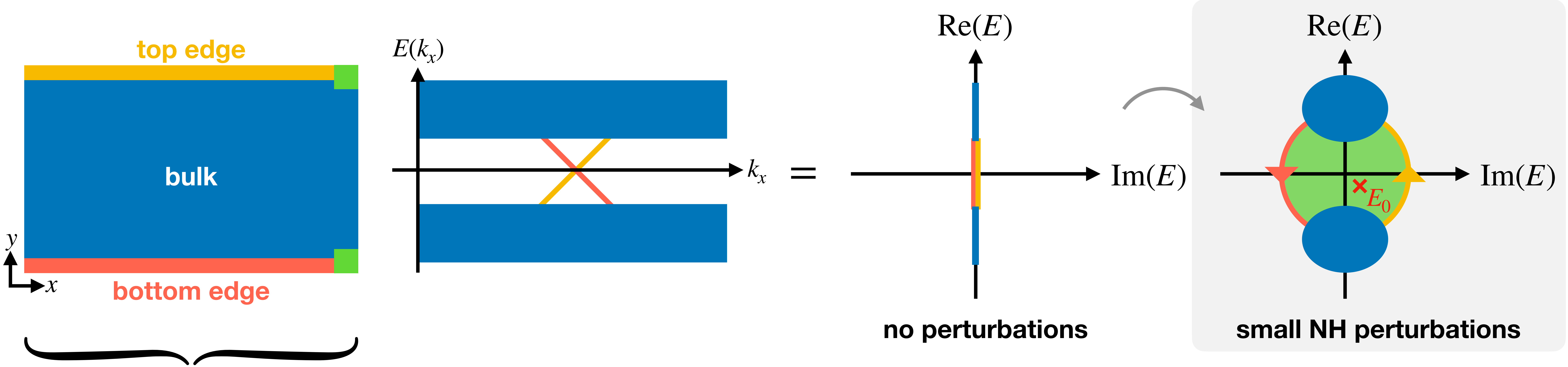

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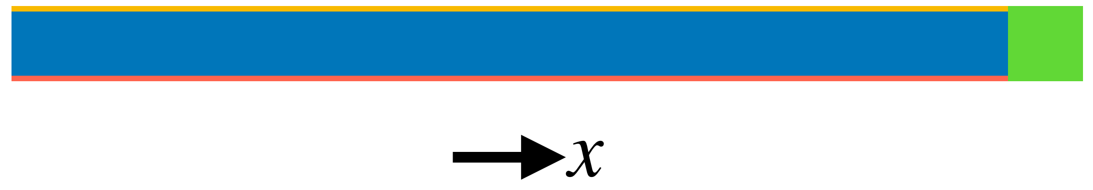

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→

higher-order
corner skin effect

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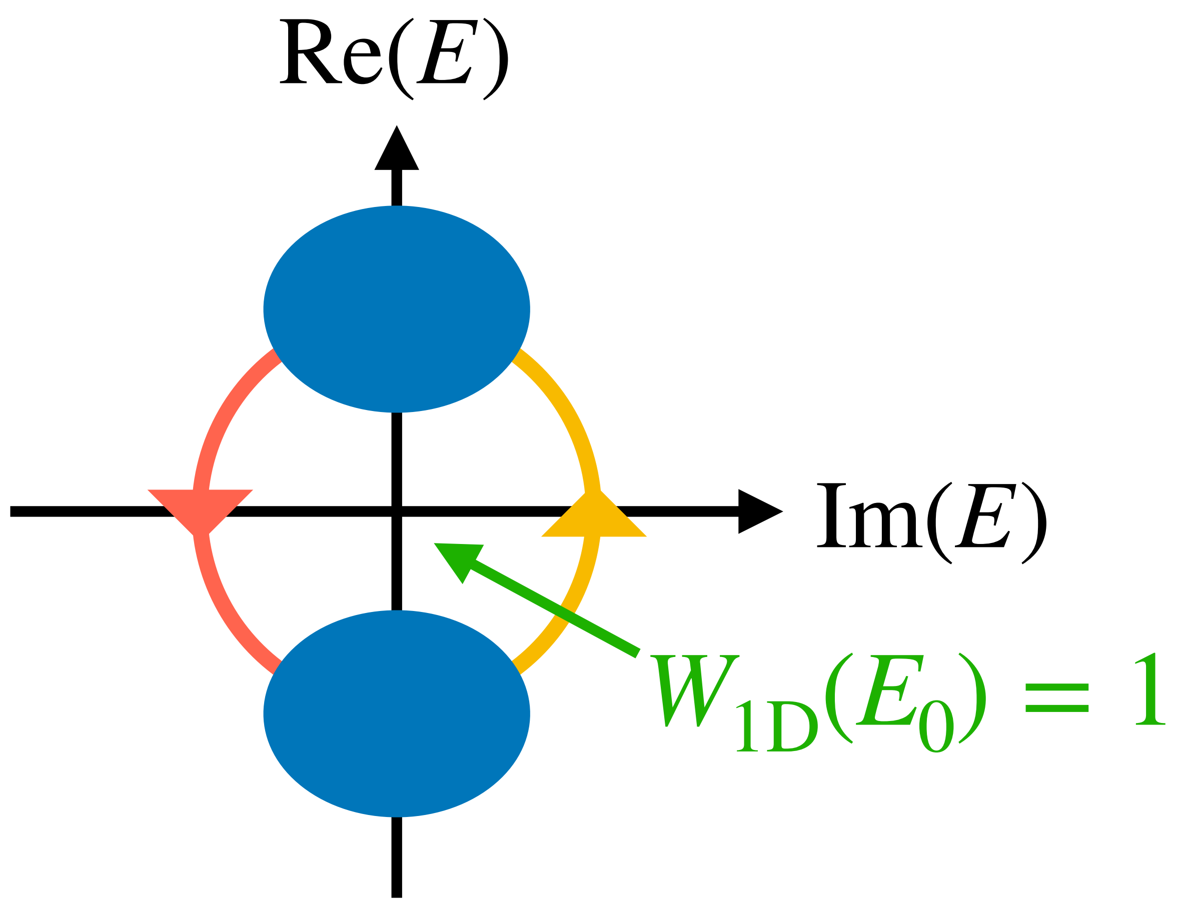
Instead of



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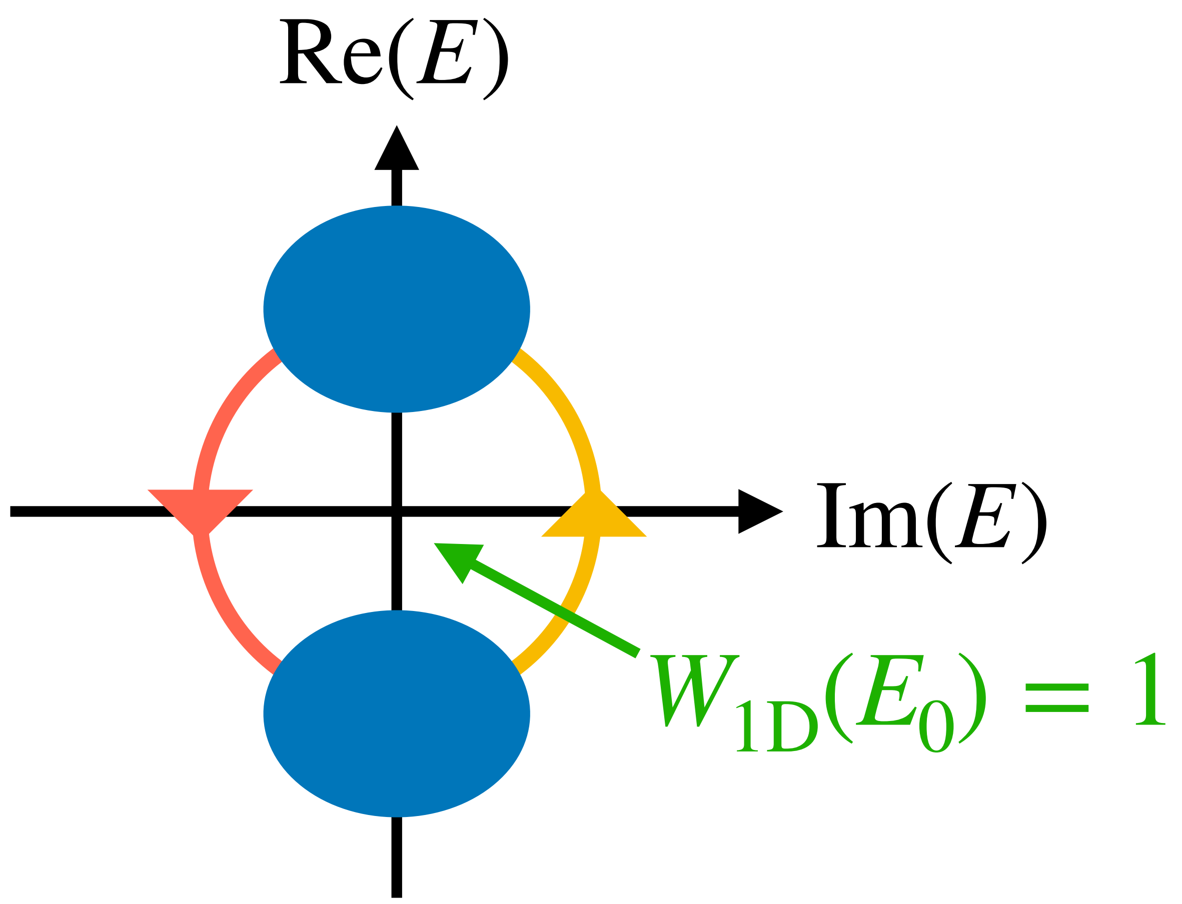
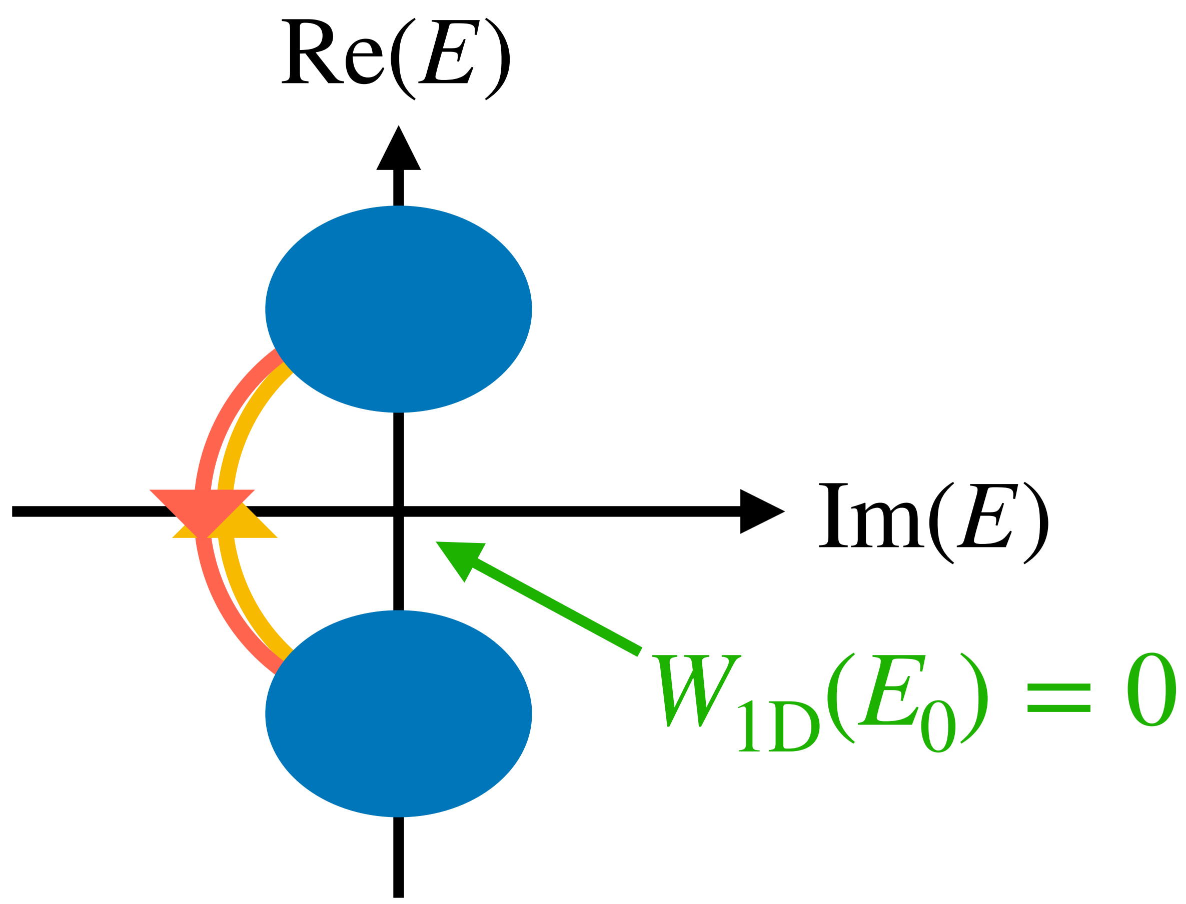


we could also have



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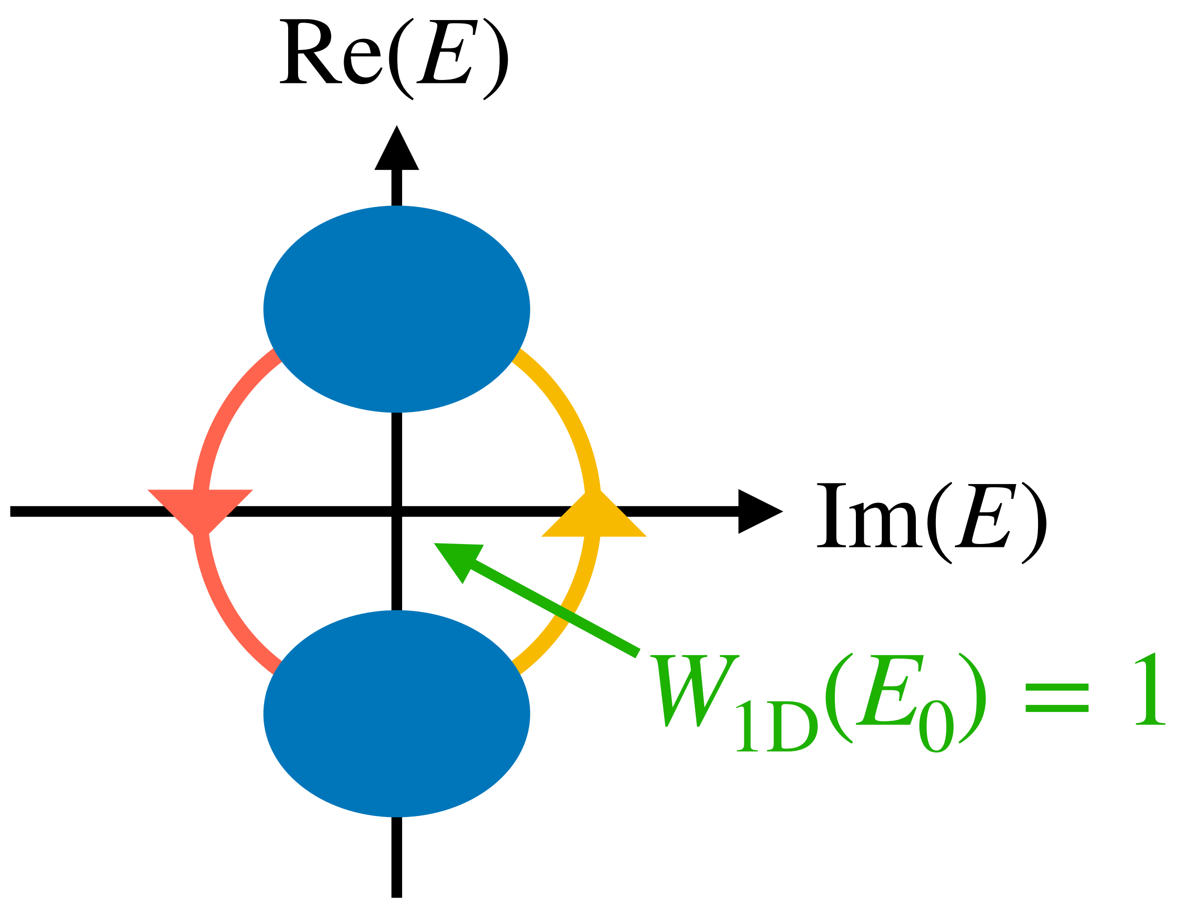

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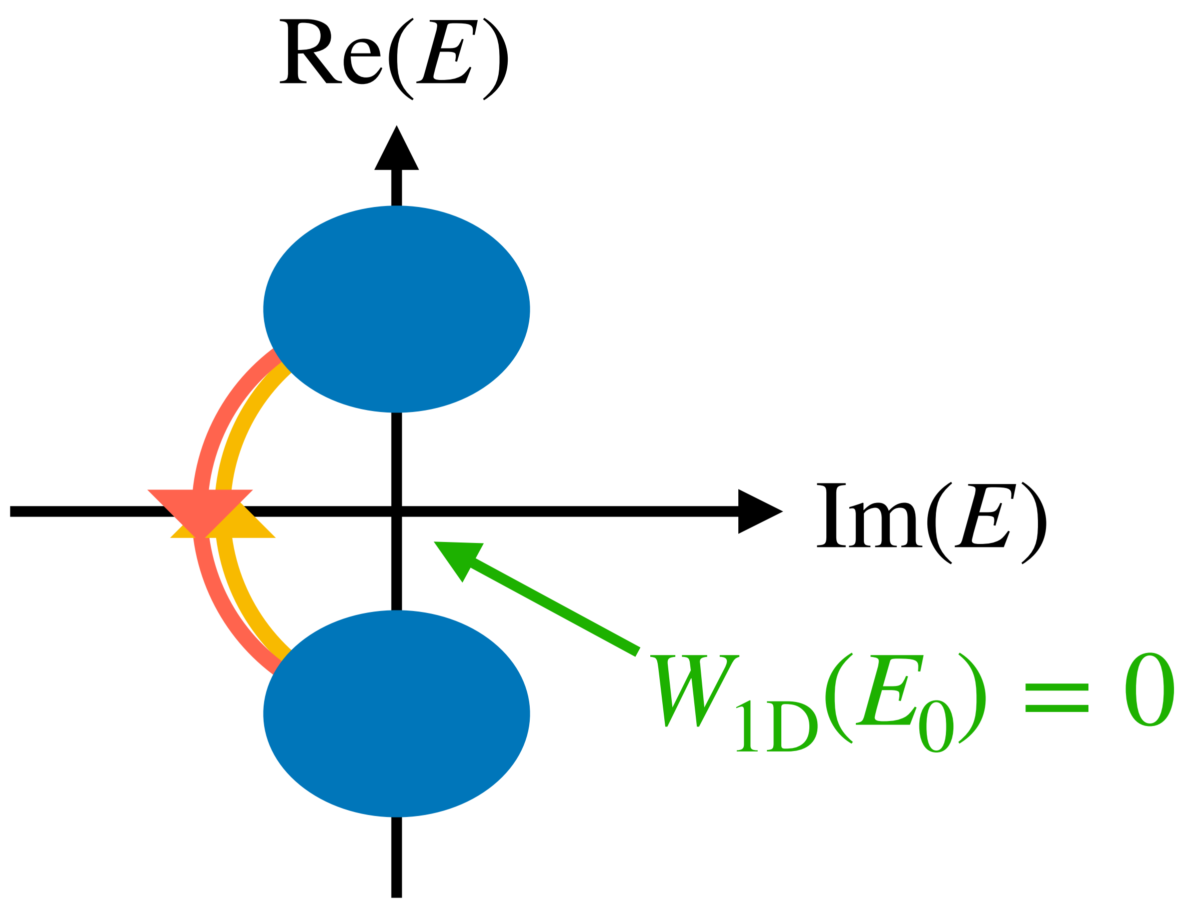

To **protect** nontrivial topology, we need a **symmetry**!

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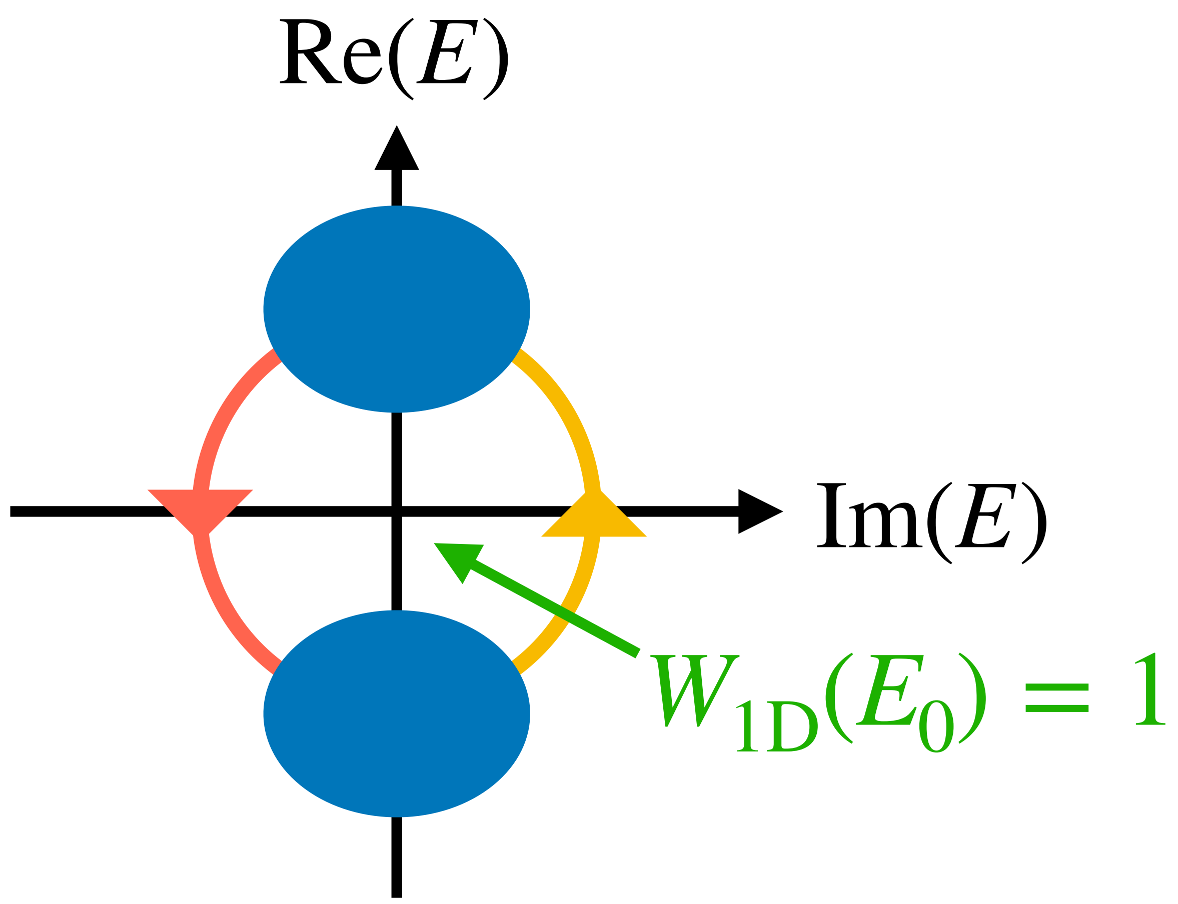

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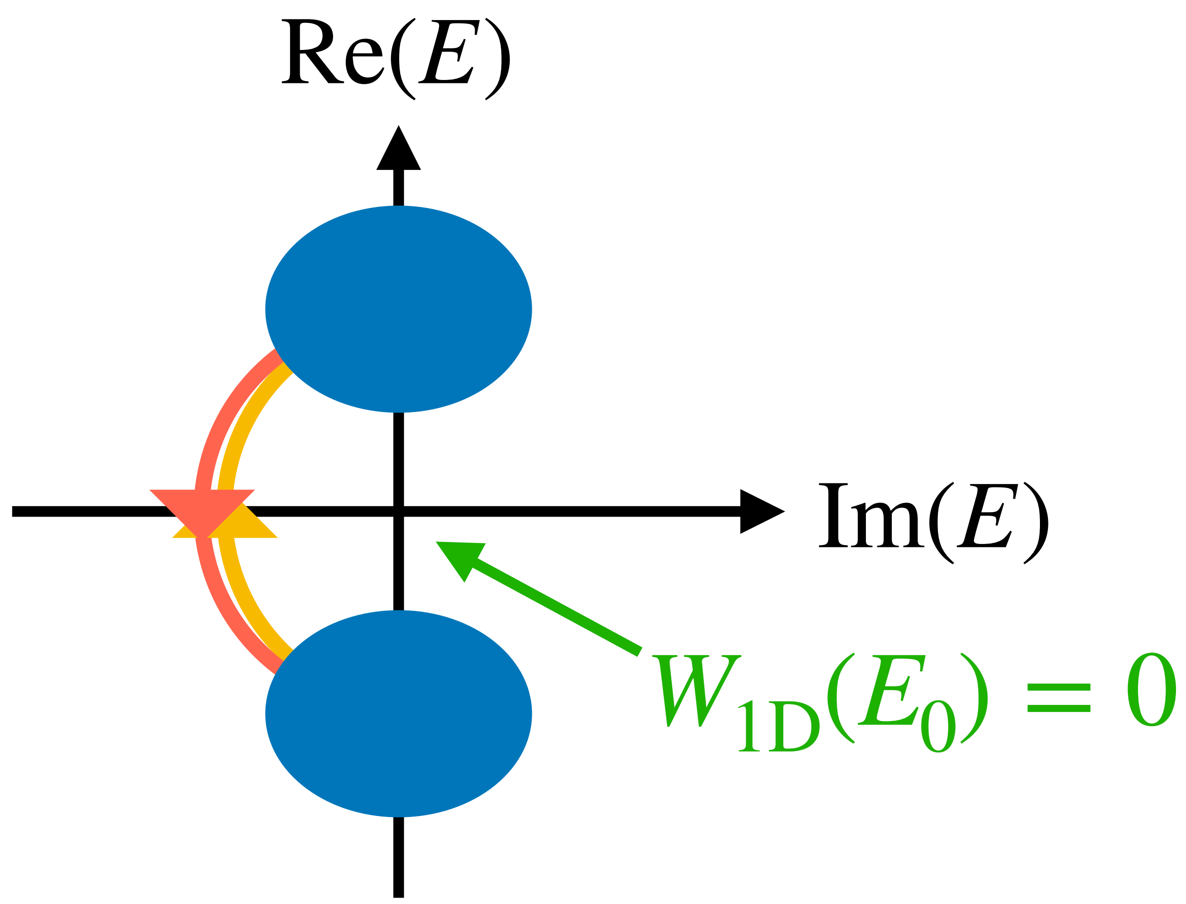

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Inversion symmetry $\mathcal{I} H(\boldsymbol{k}) \mathcal{I}^\dagger = H(-\boldsymbol{k})$

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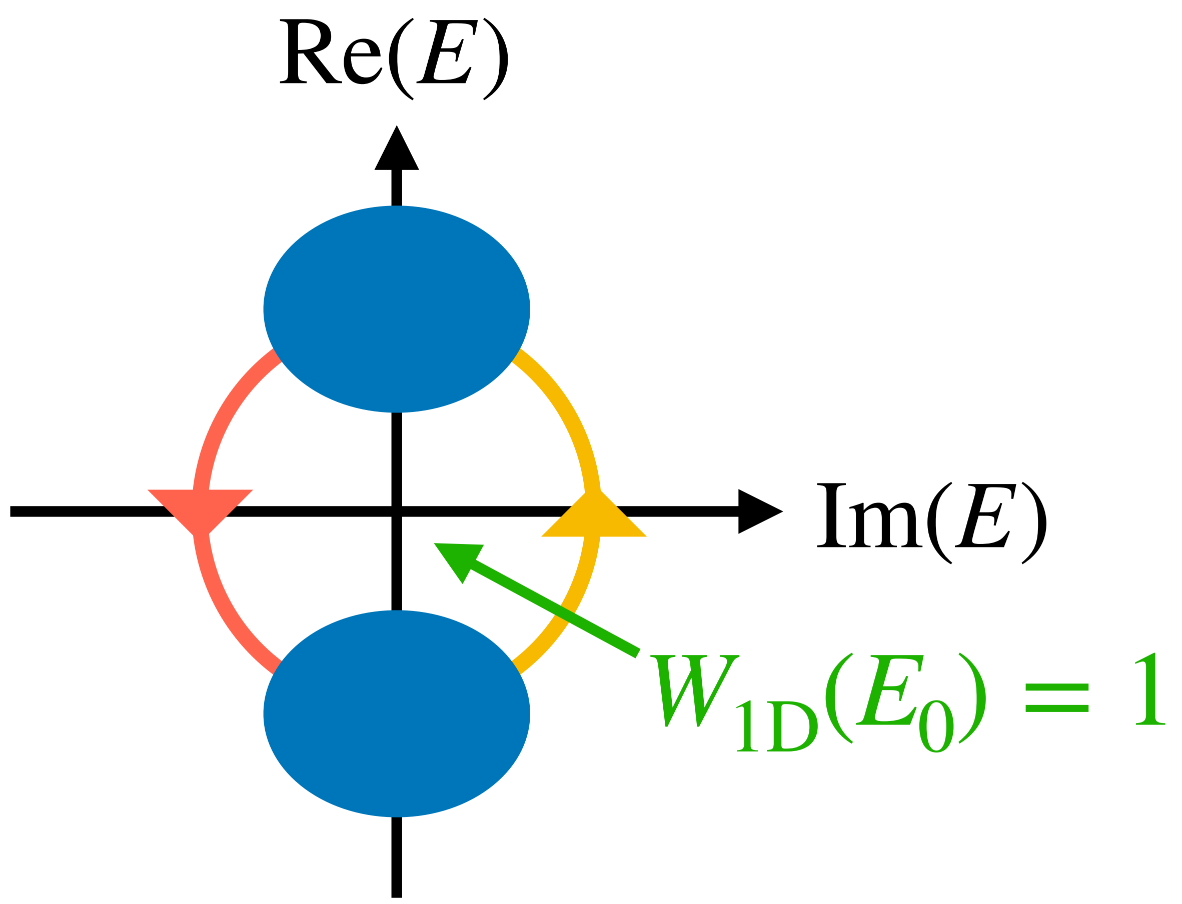

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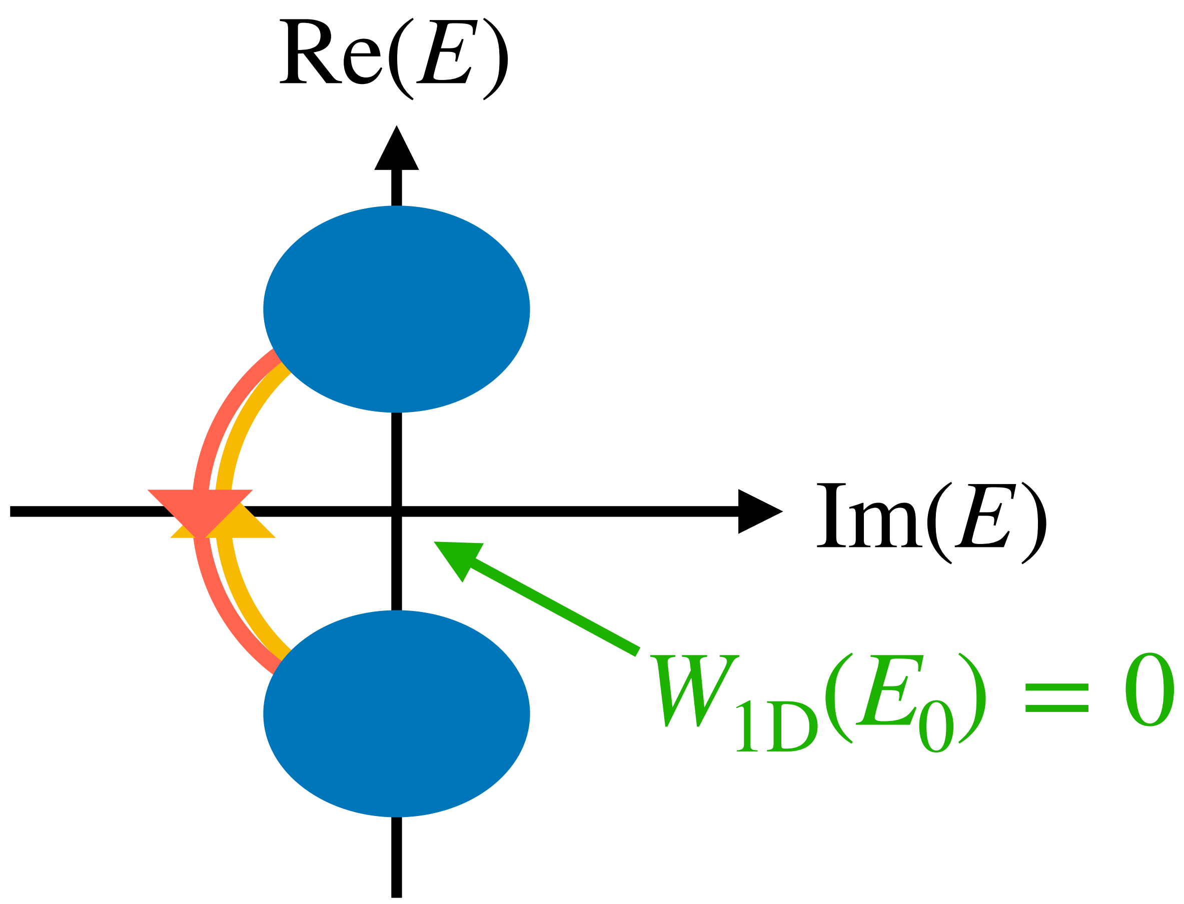

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$$\downarrow$$

$W_{1\mathrm{D}}(E_0) = 0$ 😕

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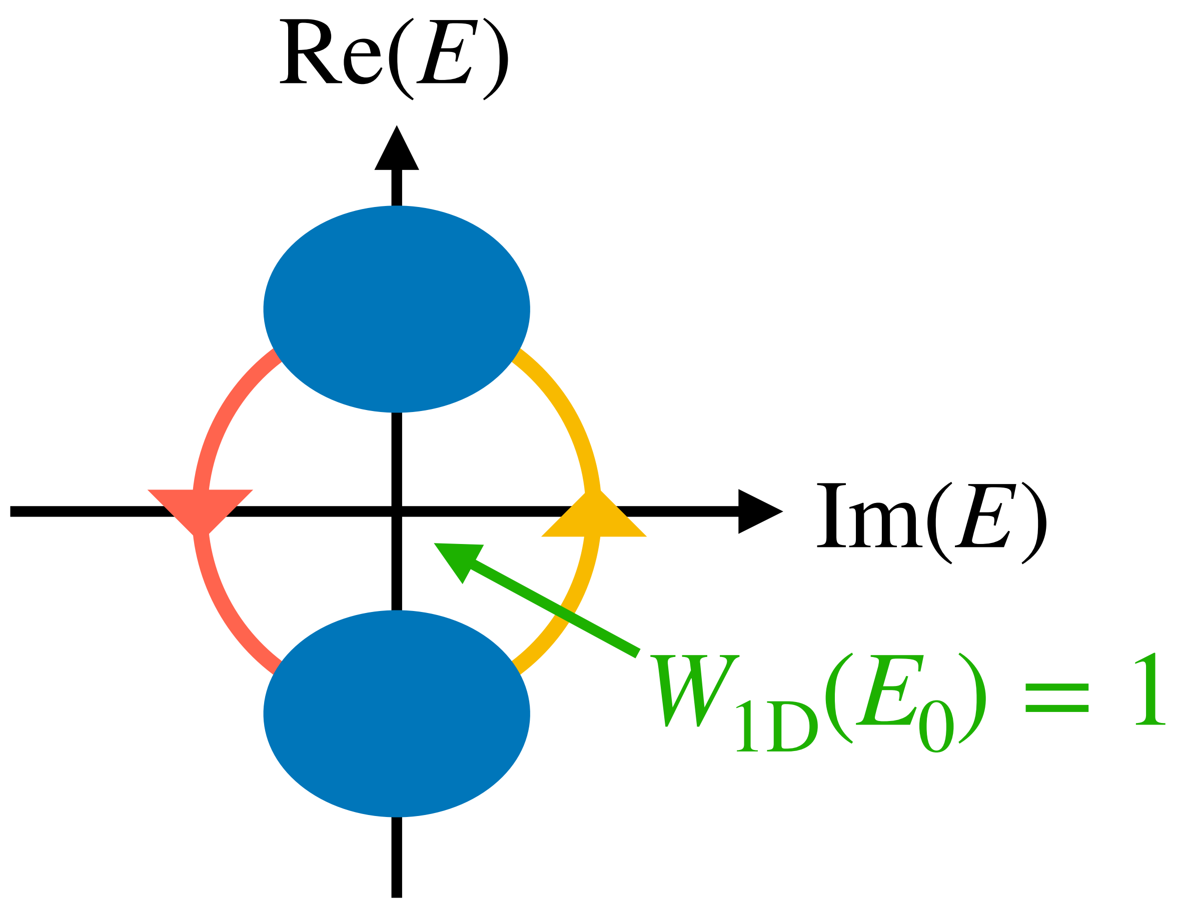

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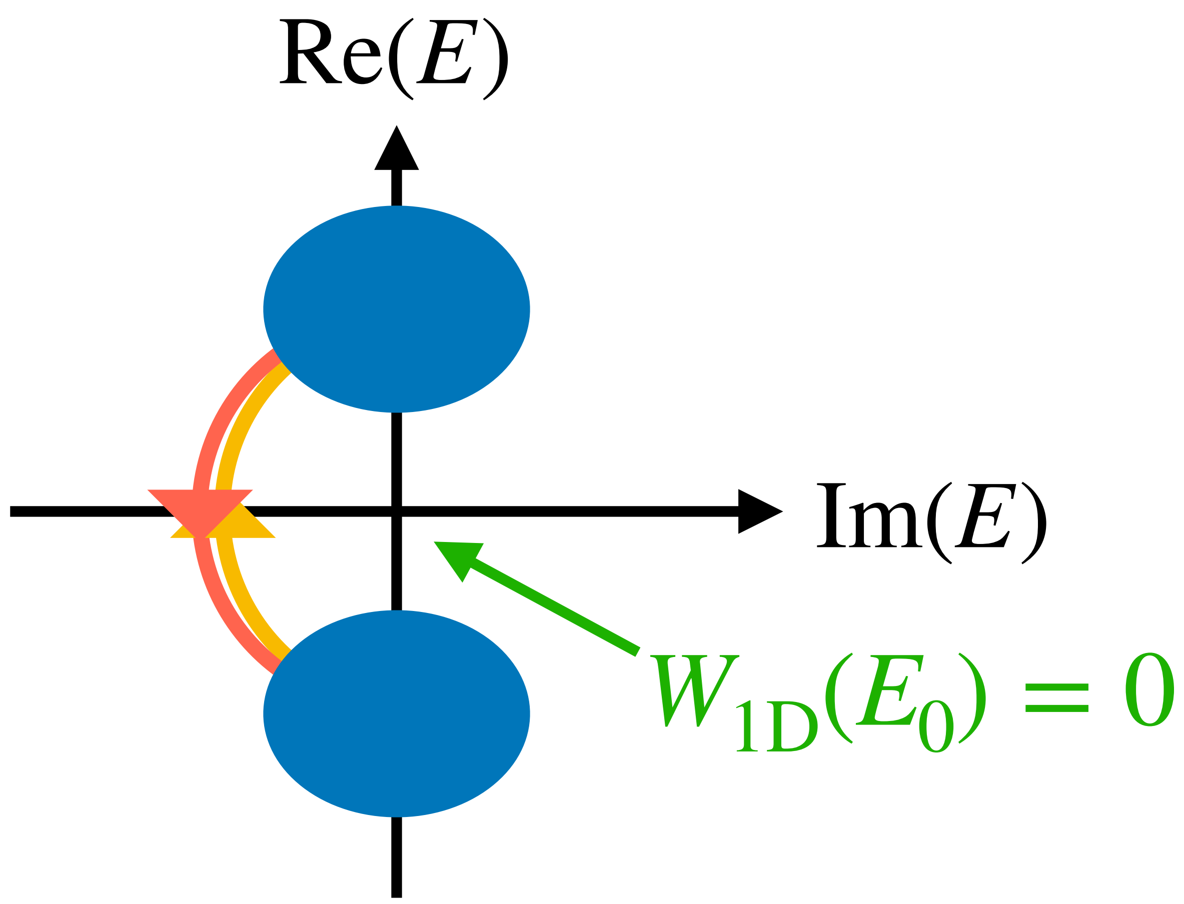

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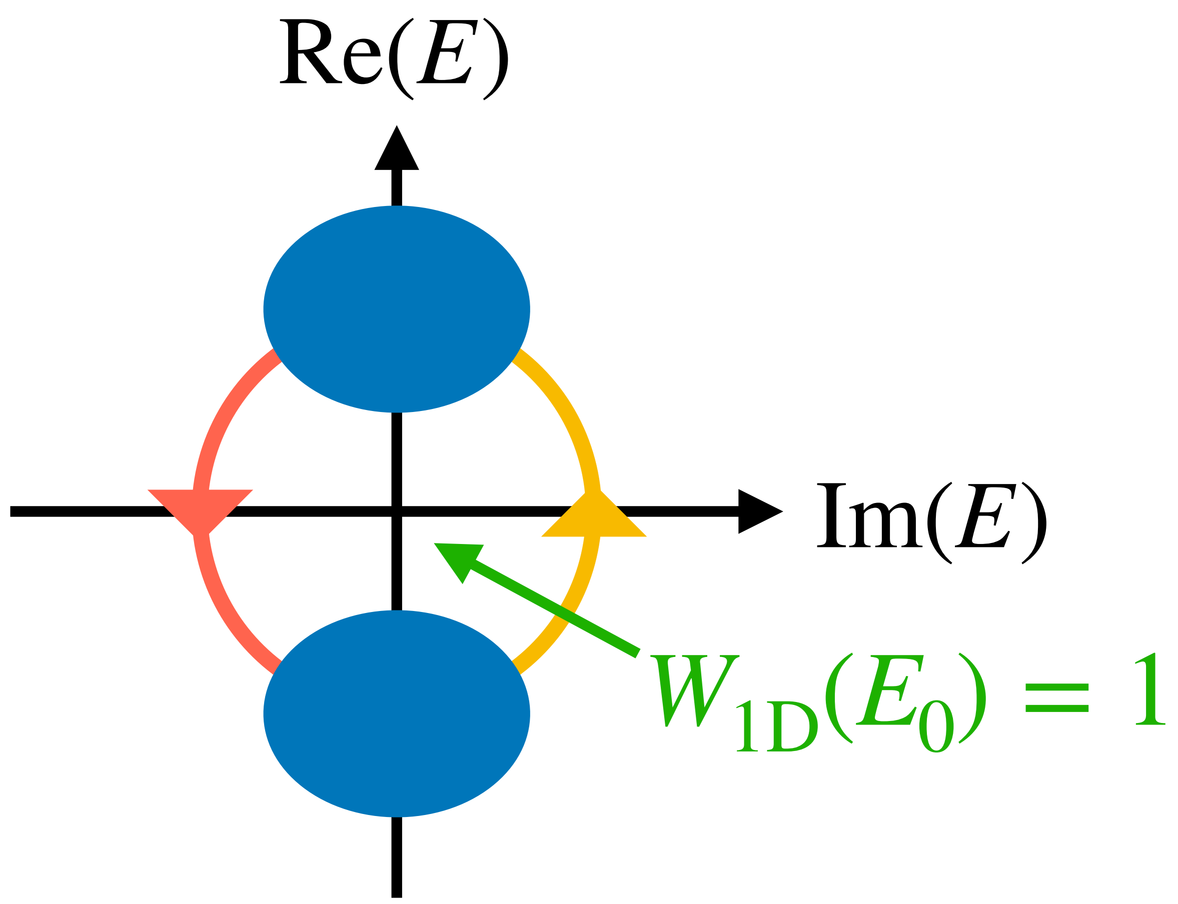
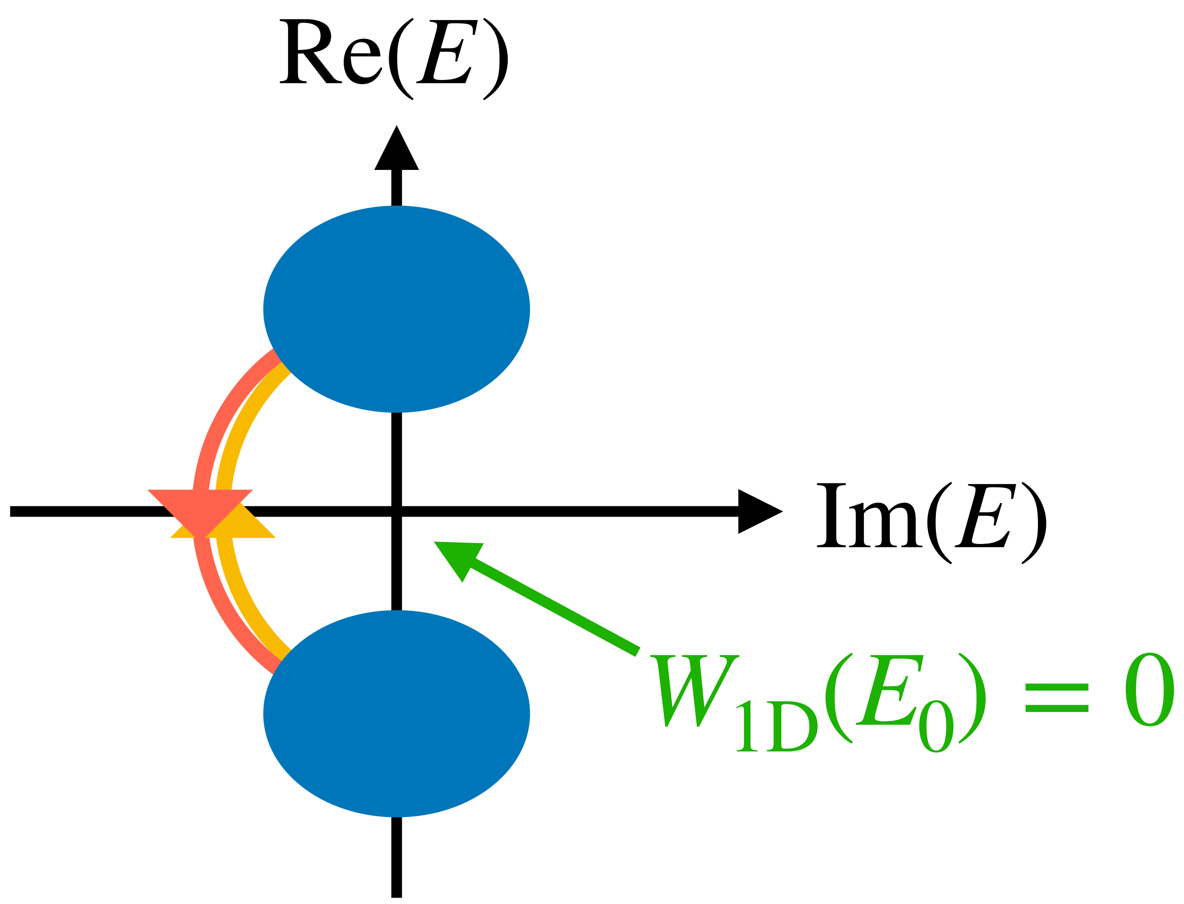
Hermitian—Non-Hermitian Correspondence

Instead of



we could also have



To **protect** nontrivial topology, we need a **symmetry**!

Inversion symmetry $\mathscr{I} H(\boldsymbol{k}) \mathscr{I}^{\dagger} = H(-\boldsymbol{k})$ —> **non-Hermitian inversion** symmetry $\mathscr{I} H(\boldsymbol{k}) \mathscr{I}^{\dagger} = H^{\dagger}(-\boldsymbol{k})$

$$\downarrow$$

$W_{1\mathrm{D}}(E_0) = 0$ 😕

$$\downarrow$$

$W_{1\mathrm{D}}(E_0) \in \mathbb{Z}$ 😀

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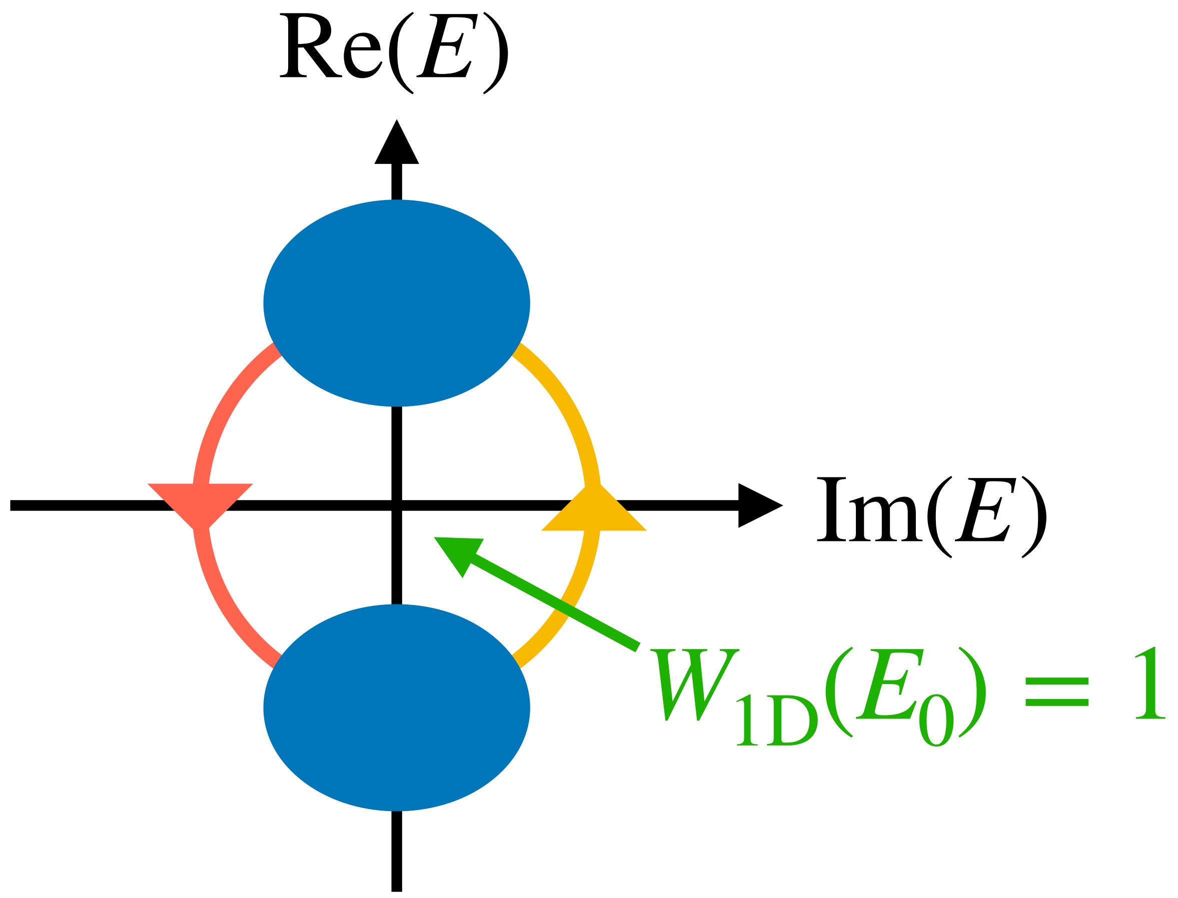

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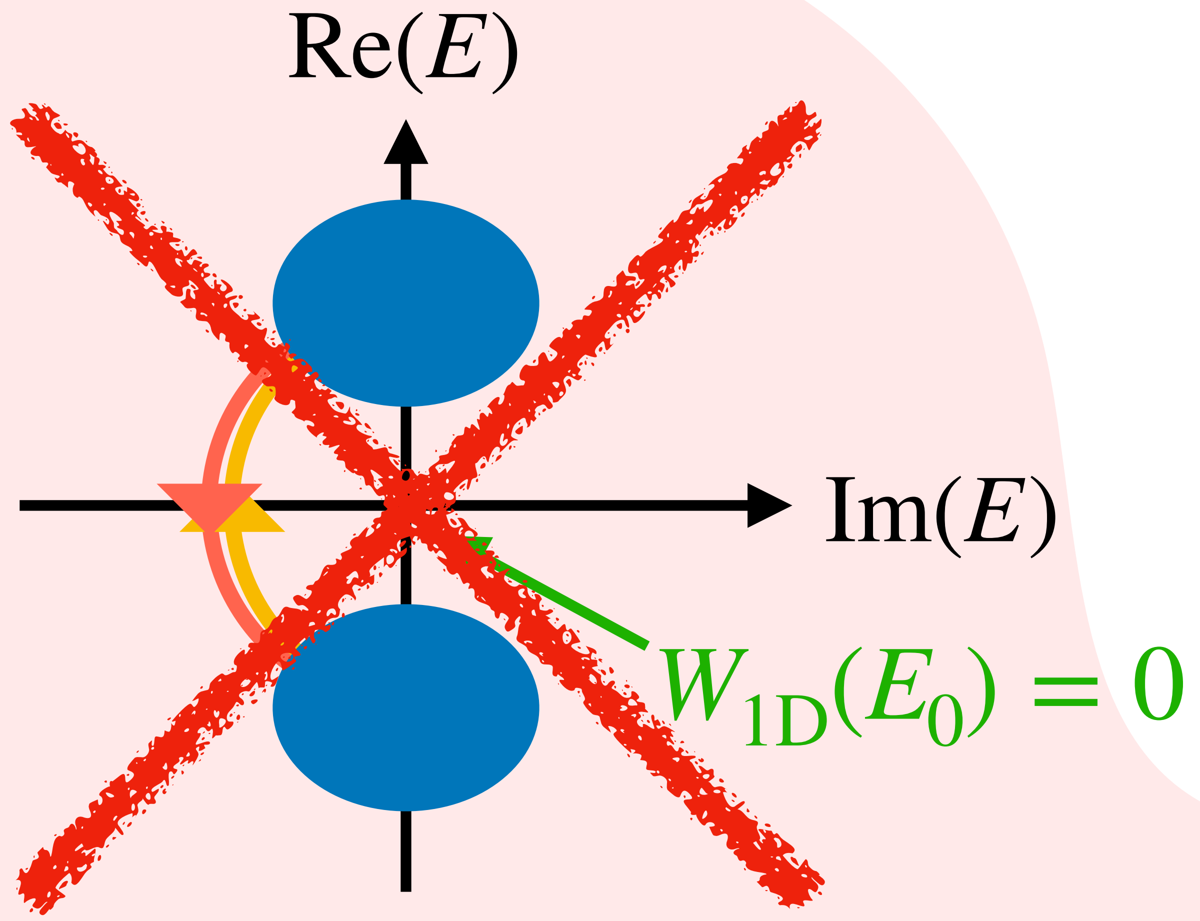

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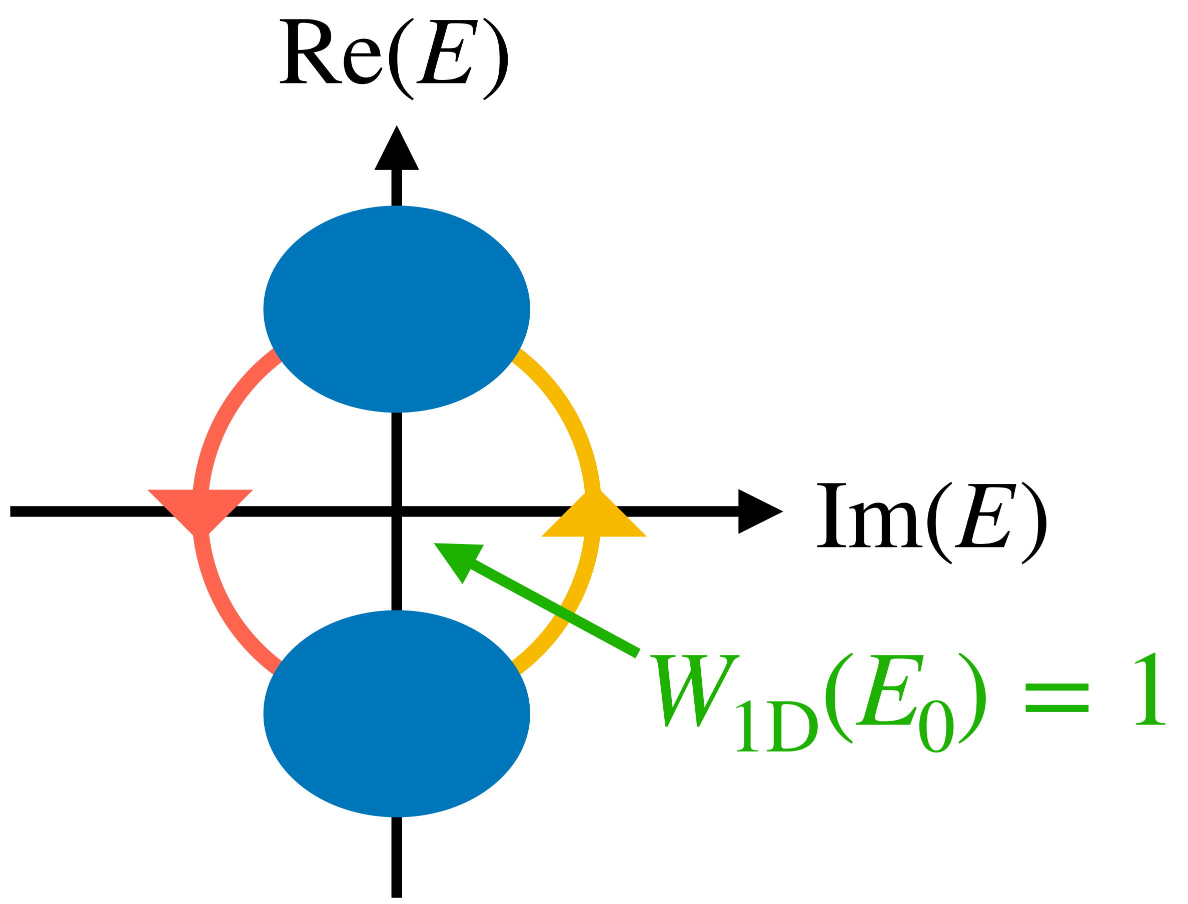

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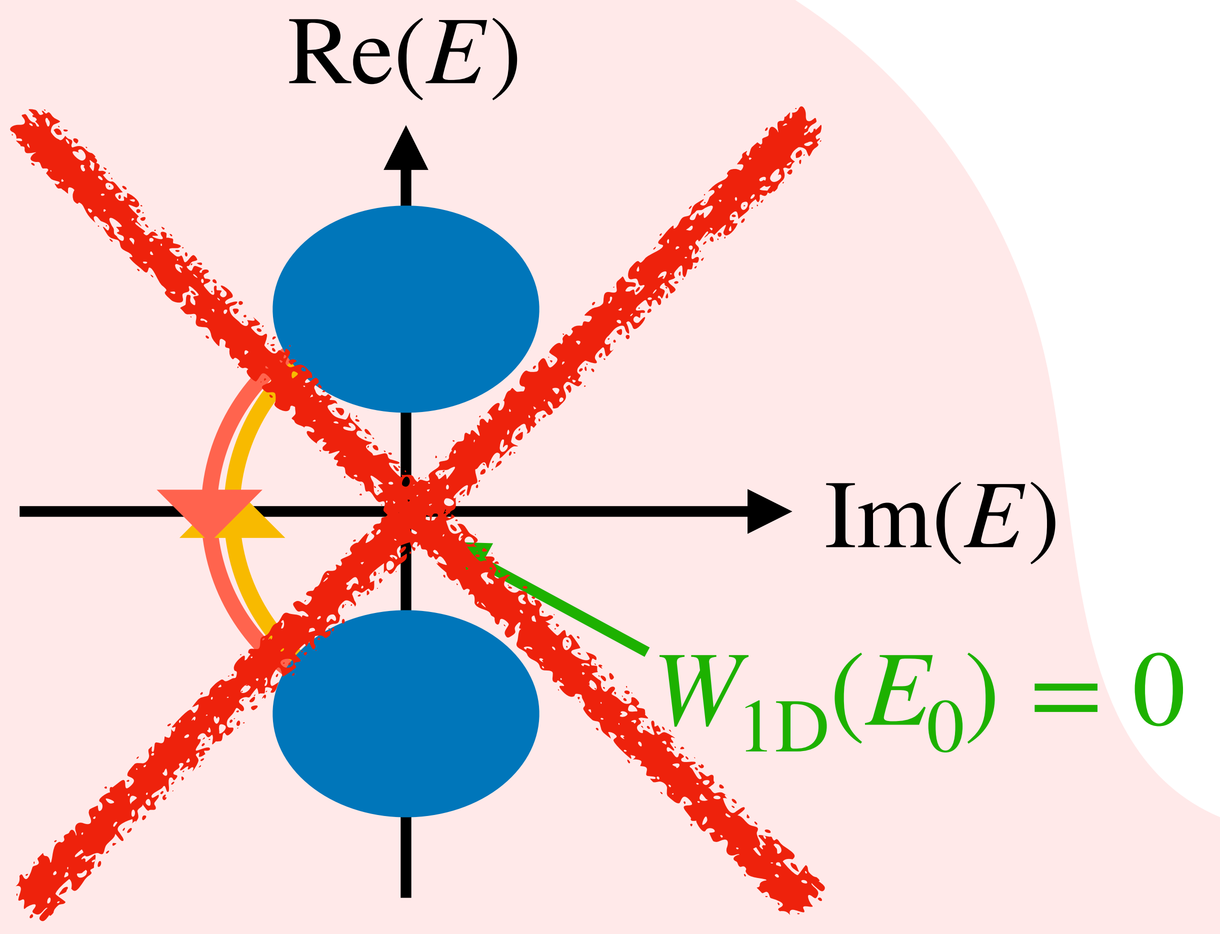

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$\downarrow$ $W_{1D}(E_0) = 0$ 😕

$\downarrow$ $W_{1D}(E_0) \in \mathbb{Z}$ 😀

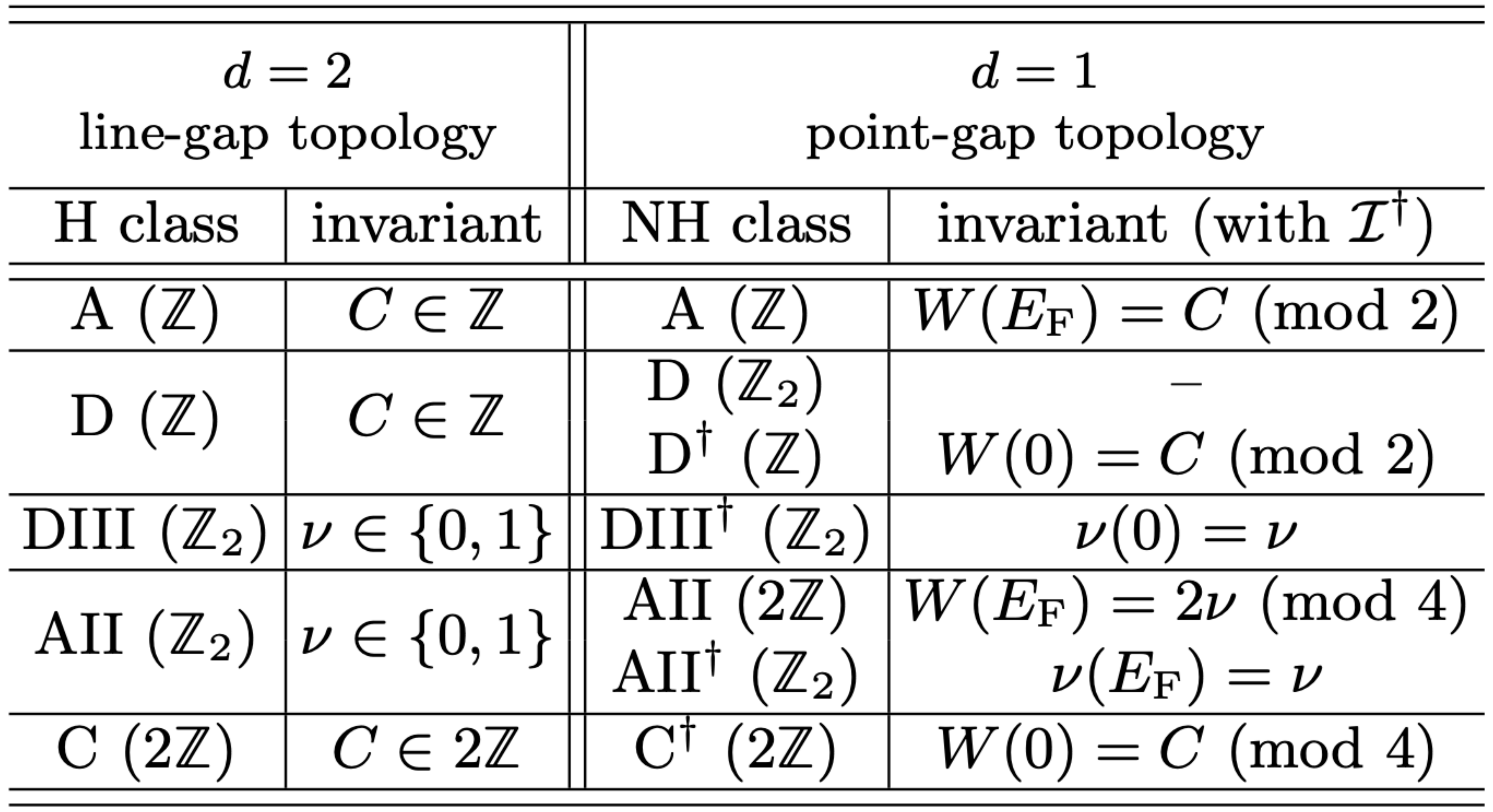
Hermitian Bulk — **Non-Hermitian** boundary correspondence:

$$\boxed{C_{2D} = W_{1D}(E_0) \mod 2}$$

Hermitian bulk invariant

Non-Hermitian boundary invariant

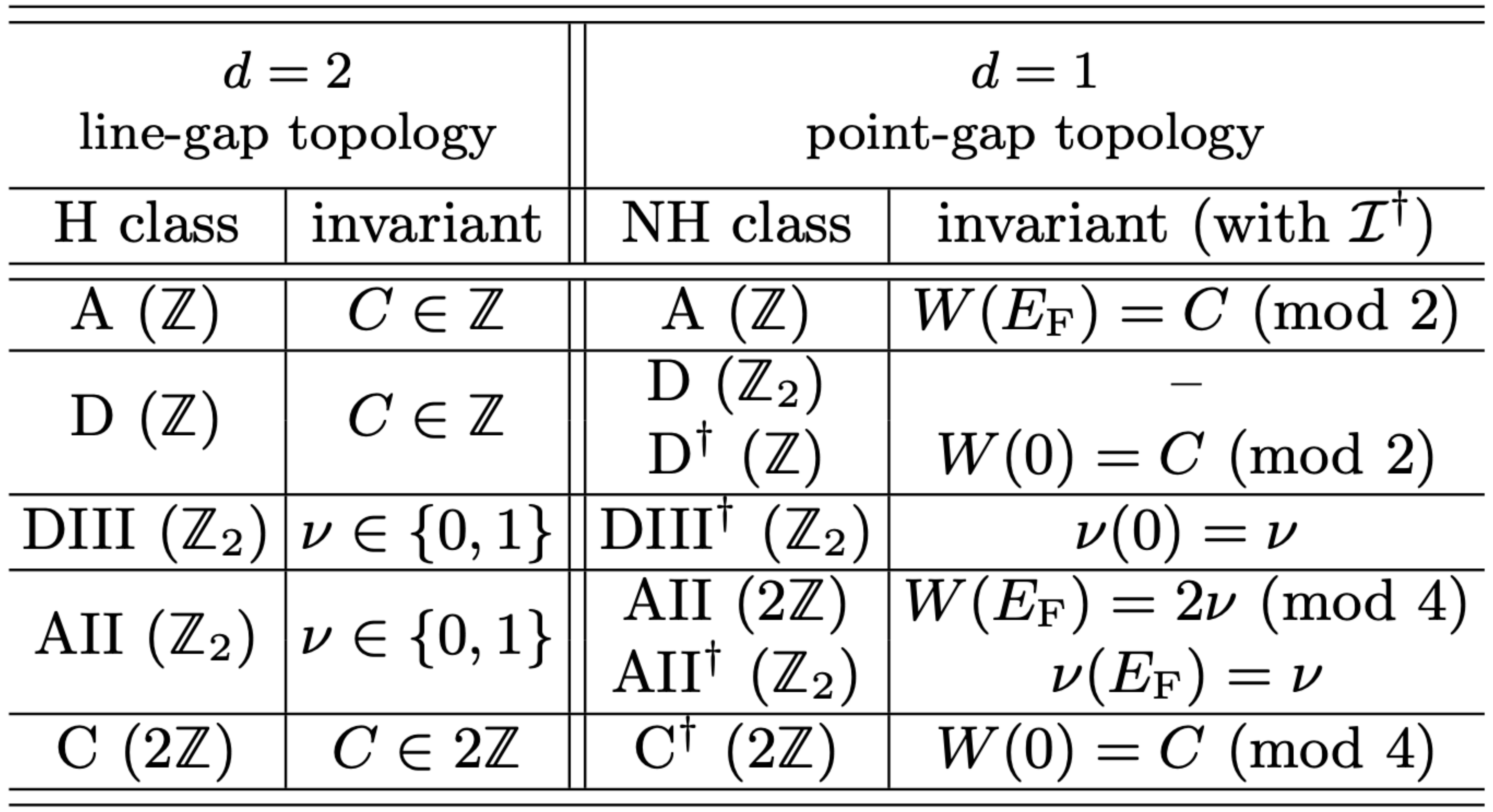
Tenfold way + NH inversion symmetry



$d=2$ line-gap topology		$d=1$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
A ($\mathbb{Z}$)	$C \in \mathbb{Z}$	A ($\mathbb{Z}$)	$W(E_\mathrm{F}) = C \pmod 2$
D ($\mathbb{Z}$)	$C \in \mathbb{Z}$	D ($\mathbb{Z}_2$) $\mathrm{D}^\dagger$ ($\mathbb{Z}$)	– $W(0) = C \pmod 2$
DIII ($\mathbb{Z}_2$)	$\nu \in \{0, 1\}$	$\mathrm{DIII}^\dagger$ ($\mathbb{Z}_2$)	$\nu(0) = \nu$
AII ($\mathbb{Z}_2$)	$\nu \in \{0, 1\}$	AII ($2\mathbb{Z}$) $\mathrm{AII}^\dagger$ ($\mathbb{Z}_2$)	$W(E_\mathrm{F}) = 2\nu \pmod 4$ $\nu(E_\mathrm{F}) = \nu$
C ($2\mathbb{Z}$)	$C \in 2\mathbb{Z}$	$\mathrm{C}^\dagger$ ($2\mathbb{Z}$)	$W(0) = C \pmod 4$

Tenfold way + NH inversion symmetry

Chern insulator →



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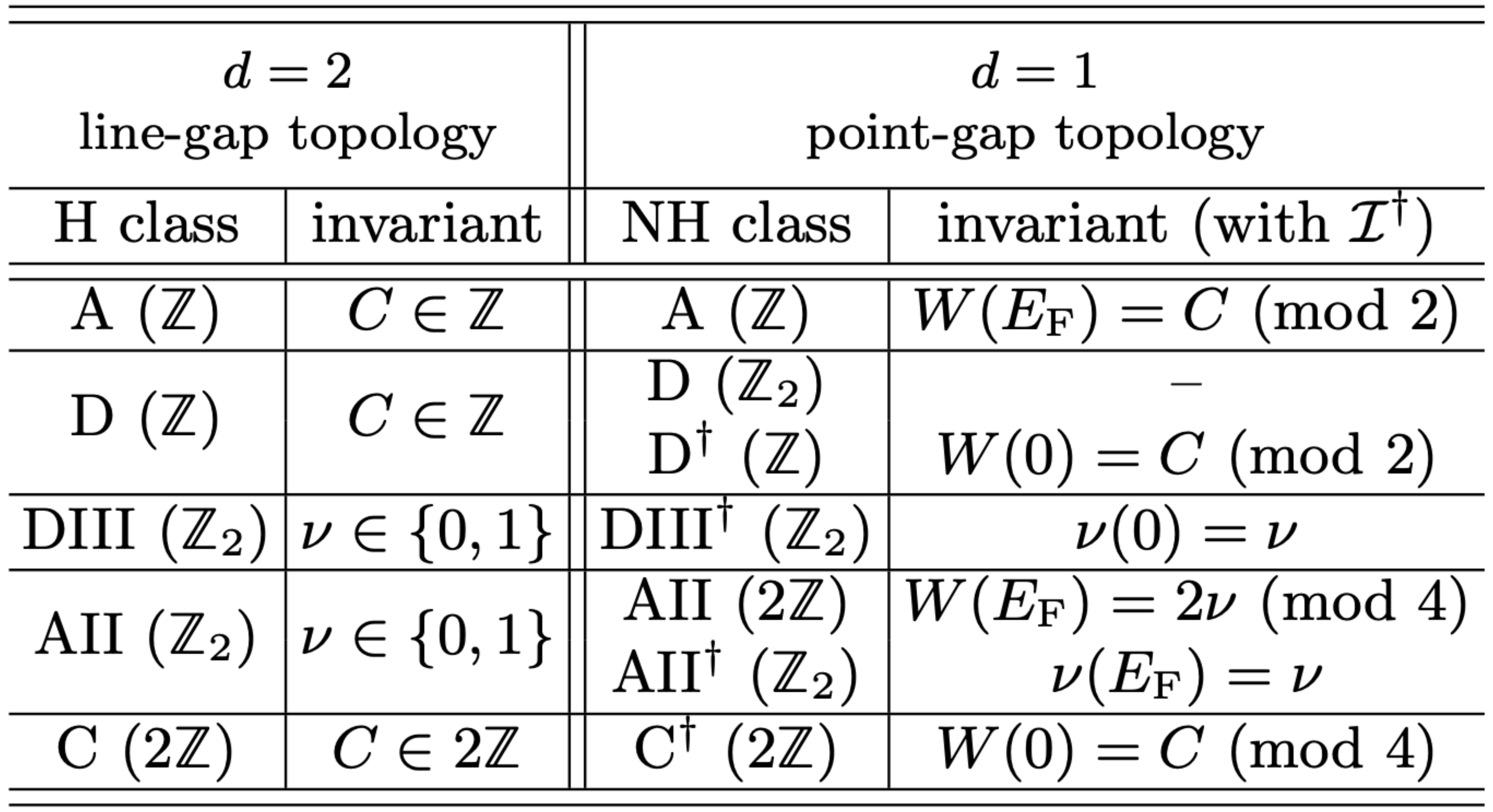
Tenfold way + NH inversion symmetry

Chern insulator →

$p_x + ip_y$ →

topological **superconductor**



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$\}$ ← $W_{1\mathrm{D}} = 1$ NH **skin effect**

Tenfold way + NH inversion symmetry

Chern insulator →

$p_x + ip_y$ topological **superconductor** →

2D **quantum spin Hall** effect/ topological insulator →

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← $W_{\mathrm{1D}} = 1$ NH **skin effect**

Tenfold way + NH inversion symmetry

Chern insulator

$p_x + ip_y$

topological **superconductor**

2D **quantum spin Hall** effect/ topological insulator

$d=2$ line-gap topology		$d=1$ point-gap topology	
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$W_{1\mathrm{D}} = 1$ NH **skin effect**

$W_{1\mathrm{D}} = 2$ NH **skin effect**

Tenfold way + NH inversion symmetry

$d=2$ line-gap topology		$d=1$ point-gap topology	
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C $(2\mathbb{Z})$	$C \in 2\mathbb{Z}$	C† $(2\mathbb{Z})$	$W(0) = C \pmod 4$

Chern insulator → A $(\mathbb{Z})$

$p_x + ip_y$ topological **superconductor** → D $(\mathbb{Z})$

2D **quantum spin Hall** effect/ topological insulator → AII $(\mathbb{Z}_2)$

$W_{1\mathrm{D}} = 1$ NH **skin effect** ← A $(\mathbb{Z})$, D† $(\mathbb{Z})$

$W_{1\mathrm{D}} = 2$ NH **skin effect** ← AII $(2\mathbb{Z})$

$\mathbb{Z}_2$ NH **skin effect** ← AII† $(\mathbb{Z}_2)$

Tenfold way + NH inversion symmetry

Chern insulator

$p_x + ip_y$

topological **superconductor**

2D **quantum spin Hall** effect/
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$d=2$ line-gap topology		$d=1$ point-gap topology	
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$W_{1\mathrm{D}} = 1$ NH **skin effect**

$W_{1\mathrm{D}} = 2$ NH **skin effect**

$\mathbb{Z}_2$ NH **skin effect**

$d=3$ line-gap topology		$d=2$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W \pmod 2$
DIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	DIII ($\mathbb{Z}_2$) DIII† ($\mathbb{Z}$)	– $C(0) = W \pmod 2$
AII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	AII† ($\mathbb{Z}_2$)	$\nu(E_\mathrm{F}) = \nu$
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CI ($2\mathbb{Z}$)	$W \in 2\mathbb{Z}$	CI† ($2\mathbb{Z}$)	$C(0) = W \pmod 4$

Tenfold way + NH inversion symmetry

Chern insulator →

$p_x + ip_y$ →
topological **superconductor**

2D **quantum spin Hall** effect/ →
topological insulator

$d=2$ line-gap topology		$d=1$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
A ($\mathbb{Z}$)	$C \in \mathbb{Z}$	A ($\mathbb{Z}$)	$W(E_\mathrm{F}) = C \pmod 2$
D ($\mathbb{Z}$)	$C \in \mathbb{Z}$	D ($\mathbb{Z}_2$) D† ($\mathbb{Z}$)	– $W(0) = C \pmod 2$
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← $W_{1\mathrm{D}} = 1$ NH **skin effect**

← $W_{1\mathrm{D}} = 2$ NH **skin effect**

← $\mathbb{Z}_2$ NH **skin effect**

3D $\mathbb{Z}$ topological **insulator** →

$d=3$ line-gap topology		$d=2$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W \pmod 2$
DIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	DIII ($\mathbb{Z}_2$) DIII† ($\mathbb{Z}$)	– $C(0) = W \pmod 2$
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CI ($2\mathbb{Z}$)	$W \in 2\mathbb{Z}$	CI† ($2\mathbb{Z}$)	$C(0) = W \pmod 4$

Tenfold way + NH inversion symmetry

Chern insulator

$p_x + ip_y$ topological **superconductor**

2D **quantum spin Hall** effect/ topological insulator

$d=2$ line-gap topology		$d=1$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
A ($\mathbb{Z}$)	$C \in \mathbb{Z}$	A ($\mathbb{Z}$)	$W(E_\mathrm{F}) = C \pmod 2$
D ($\mathbb{Z}$)	$C \in \mathbb{Z}$	D ($\mathbb{Z}_2$) D† ($\mathbb{Z}$)	– $W(0) = C \pmod 2$
DIII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	DIII† ($\mathbb{Z}_2$)	$\nu(0) = \nu$
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C ($2\mathbb{Z}$)	$C \in 2\mathbb{Z}$	C† ($2\mathbb{Z}$)	$W(0) = C \pmod 4$

$W_{\mathrm{1D}} = 1$ NH **skin effect**

$W_{\mathrm{1D}} = 2$ NH **skin effect**

$\mathbb{Z}_2$ NH **skin effect**

3D $\mathbb{Z}$ topological **insulator**

3D $\mathbb{Z}$ topological **superconductor**

$d=3$ line-gap topology		$d=2$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W \pmod 2$
DIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	DIII ($\mathbb{Z}_2$) DIII† ($\mathbb{Z}$)	– $C(0) = W \pmod 2$
AII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	AII† ($\mathbb{Z}_2$)	$\nu(E_\mathrm{F}) = \nu$
CII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	CII ($2\mathbb{Z}$) CII† ($\mathbb{Z}_2$)	$C(0) = 2\nu \pmod 4$ $\nu(0) = \nu$
CI ($2\mathbb{Z}$)	$W \in 2\mathbb{Z}$	CI† ($2\mathbb{Z}$)	$C(0) = W \pmod 4$

Tenfold way + NH inversion symmetry

$d=2$ line-gap topology		$d=1$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^{\dagger}$)
A ($\mathbb{Z}$)	$C \in \mathbb{Z}$	A ($\mathbb{Z}$)	$W(E_{\mathrm{F}}) = C$ (mod 2)
D ($\mathbb{Z}$)	$C \in \mathbb{Z}$	D ($\mathbb{Z}_2$) D† ($\mathbb{Z}$)	– $W(0) = C$ (mod 2)
DIII ($\mathbb{Z}_2$)	$\nu \in \{0, 1\}$	DIII† ($\mathbb{Z}_2$)	$\nu(0) = \nu$
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C ($2\mathbb{Z}$)	$C \in 2\mathbb{Z}$	C† ($2\mathbb{Z}$)	$W(0) = C$ (mod 4)

Chern insulator →

$p_x + ip_y$ →

topological **superconductor**

2D **quantum spin Hall** effect/ →
topological insulator

← $W_{\mathrm{1D}} = 1$ NH **skin effect**

← $W_{\mathrm{1D}} = 2$ NH **skin effect**

← $\mathbb{Z}_2$ NH **skin effect**

$d=3$ line-gap topology		$d=2$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^{\dagger}$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W$ (mod 2)
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3D $\mathbb{Z}$ topological **insulator** →

3D $\mathbb{Z}$ topological **superconductor** →

← NH chiral **hinge modes**

Tenfold way + NH inversion symmetry

Chern insulator →

$p_x + ip_y$ →
topological **superconductor**

2D **quantum spin Hall** effect/
topological insulator →

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← $W_{\mathrm{1D}} = 1$ NH **skin effect**

← $W_{\mathrm{1D}} = 2$ NH **skin effect**

← $\mathbb{Z}_2$ NH **skin effect**

3D $\mathbb{Z}$ topological **insulator** →

3D $\mathbb{Z}$ topological **superconductor** →

3D $\mathbb{Z}_2$ topological **insulator** →

$d=3$ line-gap topology		$d=2$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W \pmod 2$
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← NH chiral **hinge modes**

Tenfold way + NH inversion symmetry

Chern insulator

$p_x + ip_y$ topological **superconductor**

2D **quantum spin Hall** effect/ topological insulator

$d=2$ line-gap topology		$d=1$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
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$W_{1\mathrm{D}} = 1$ NH **skin effect**

$W_{1\mathrm{D}} = 2$ NH **skin effect**

$\mathbb{Z}_2$ NH **skin effect**

3D $\mathbb{Z}$ topological **insulator**

3D $\mathbb{Z}$ topological **superconductor**

3D $\mathbb{Z}_2$ topological **insulator**

$d=3$ line-gap topology		$d=2$ point-gap topology	
H class	invariant	NH class	invariant (with $\mathcal{I}^\dagger$)
AIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	AIII ($\mathbb{Z}$)	$C(0) = W \pmod 2$
DIII ($\mathbb{Z}$)	$W \in \mathbb{Z}$	DIII ($\mathbb{Z}_2$) DIII† ($\mathbb{Z}$)	– $C(0) = W \pmod 2$
AII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	AII† ($\mathbb{Z}_2$)	$\nu(E_\mathrm{F}) = \nu$
CII ($\mathbb{Z}_2$)	$\nu \in \{0,1\}$	CII ($2\mathbb{Z}$) CII† ($\mathbb{Z}_2$)	$C(0) = 2\nu \pmod 4$ $\nu(0) = \nu$
CI ($2\mathbb{Z}$)	$W \in 2\mathbb{Z}$	CI† ($2\mathbb{Z}$)	$C(0) = W \pmod 4$

NH chiral **hinge modes**

2D NH skin effect probed by **π-flux**

Summary

Hermitian topological phases stabilize
intrinsically NH **boundary point gap topology**
subject to **small NH perturbations** preserving a **NH crystalline symmetry**

This gives rise to **higher-order** NH **skin effects** and **hinge modes**

There are **>6000** known topological insulator **compounds**.
All of these are now **material candidates** for realizing NH topology!

Open questions: classification for **higher symmetries**, **field theory** perspective, realistic mechanisms for generating NH perturbations (phonons, strong correlations, …?)

Thank you for your attention!

Collaborators

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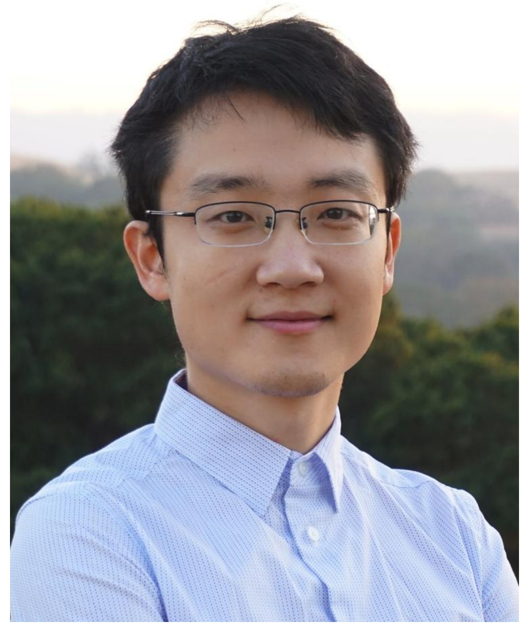

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