The geometric cobordism hypothesis

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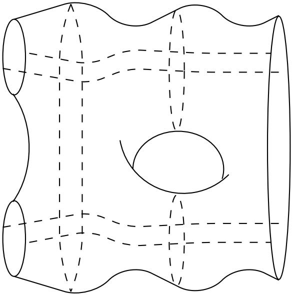
These slides: `https://dmitripavlov.org/nyuad.pdf`

arXiv:2011.01208, arXiv:2111.01095 (joint with Daniel Grady)



Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula
- Later: path integral used in QFT, no longer rigorous
- 1980s (Witten): properties of path integrals for (conformal) field theory
- 1980s (Segal): mathematical formulation of conformal field theory

Further developments

- late 1980s (Atiyah, Kontsevich, ...): topological theories: easier to construct and study, but less relevant for physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)
- 1995 (Baez–Dolan): the topological cobordism and tangle hypotheses
- 2002 (Stolz–Teichner): modern formulation of nontopological field theories (including supersymmetry); the Stolz–Teichner program on $2|1$-EFTs and TMF
- 2004 (Costello): the $(\infty, 2)$-category of topological 2-dimensional bordisms
- 2006 (Hopkins–Lurie); 2015 (Calaque–Scheimbauer): the (∞, d)-category of topological bordisms

Previous results on the topological cobordism hypothesis

- 2008 (Lurie): outline of a proof of the topological cobordism hypothesis
- 2017 (Ayala–Francis): a different approach, conditional on a conjecture
- 2004 (Costello), 2009 (Schommer-Pries): the 2-dimensional topological cobordism hypothesis
- 2006 (Galatius–Madsen–Tillmann–Weiss); 2011 (Bökstedt–Madsen); 2017 (Schommer-Pries): the invertible case

Low-dimensional nontopological field theories

Examples of 2-dimensional nonextended nontopological field theories:

- 2007 (Pickrell): Riemannian 2-dimensional field theory
- 2018 (Runkel–Szegedy): volume-dependent 2-dimensional field theory

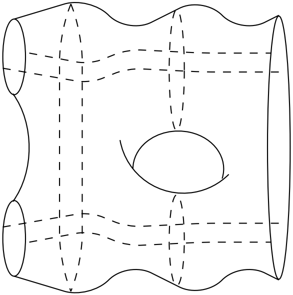
Classifications of holonomy maps, transport functors, and 1-dimensional nontopological field theories:

- 1990 (Barrett), 1994 (Caetano–Picken), 2007 (Schreiber–Waldorf): parallel transport for bundles
- 2000 (Mackaay–Picken), 2004 (Picken), 2008 (Schreiber–Waldorf): parallel transport for gerbes
- 2015 (Berwick-Evans–P.), 2020 (Ludewig–Stoffel): 1-dimensional field theories

Features of the geometric bordism category

- Locality: k-bordisms with corners of all codimensions (up to d) with compositions in d directions
  $\implies$ symmetric monoidal d-category of bordisms
- Isotopy: chain complexes to encode BV-BRST
  $\implies$ must encode (higher) diffeomorphisms between bordisms
  $\implies$ symmetric monoidal (∞, d)-categories
- Geometric (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic/contact structures,
  principal G-bundles with connection and isos,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  $\implies$ an $(\infty, 1)$-sheaf of geometric structures
- Smoothness: values of field theories depend smoothly on bordisms
  $\implies$ $(\infty, 1)$-sheaf of (∞, d)-categories of bordisms

How to compose bordisms



Geometric structures

Definition

Given $d \geq 0$, the site FEmb_d has

- Objects: submersions $T \to U$ with d-dimensional fibers, where $U \cong \mathbf{R}^n$ is a cartesian manifold;
- Morphisms: commutative squares with $T \to T'$ a fiberwise open embedding over a smooth map $U \to U'$;
- Covering families: open covers on total spaces T.

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Definition

Given $d \geq 0$, a d-dimensional geometric structure is a simplicial presheaf $\mathcal{S}\colon \mathsf{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$.

Example:

- $T \to U \mapsto$ the set of fiberwise Riemannian metrics on $T \to U$;
- $(T \to T', U \to U') \mapsto$ the restriction map from T' to T.

Examples of geometric structures

- fiberwise Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- fiberwise conformal, complex, symplectic, contact, Kähler structures;
- fiberwise foliations, possibly with transversal metrics;
- smooth map to a target manifold M (traditional σ-model);
- smooth map to an orbifold or ∞-sheaf on manifolds;
- fiberwise etale map or an open embedding into a target manifold N;
- fiberwise topological structures: orientation, framing, etc.
- fiberwise differential n-forms (possibly closed).

Examples of geometric structures: gauge transformations

Definition

- Send a d-manifold M to (the nerve of) the groupoid $\mathrm{B}_\nabla G(M)$:
  - Objects: principal G-bundles on T with a fiberwise connection on $T \to U$ (gauge fields);
  - Morphisms: connection-preserving isomorphisms (gauge transformations).

Examples of geometric structures: (higher) gauge transformations

- Principal G-bundles with connection on M (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on M (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on M (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on M (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle d-bundles).
- Geometric tangential structures: geometric Spin^c-structure, String (Waldorf), Fivebrane (Sati–Schreiber–Stasheff), Ninebrane (Sati). (Vanishing of anomaly.)
- differential K-theory (Ramond–Ramond field). Requires ∞-groupoids.

The main theorem

Ingredients:

- A dimension $d \geq 0$.
- A smooth symmetric monoidal (∞, d)-category $\mathcal{V}$ of values.
- A d-dimensional geometric structure $\mathcal{S}\colon \mathsf{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$.

Constructions:

- The smooth symmetric monoidal (∞, d)-category of bordisms $\mathfrak{Bord}_d^{\mathcal{S}}$ with geometric structure $\mathcal{S}$.
- A d-dimensional functorial field theory valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is a smooth symmetric monoidal (∞, d)-functor $\mathfrak{Bord}_d^{\mathcal{S}} \to \mathcal{V}$.
- The simplicial set of d-dimensional functorial field theories valued in $\mathcal{V}$ with geometric structure $\mathcal{S}$ is the derived mapping simplicial set

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbf{R}\,\mathrm{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}).$$

Can be refined to a derived internal hom.

The main theorem

Conjectures:

- Freed, Lawrence (1992): $\mathsf{FFT}_{d,\mathcal{V}}$ is an ∞-sheaf.
- Baez–Dolan (1995), Hopkins–Lurie (2008): if $\mathcal{V}$ is fully dualizable,

$$\mathsf{FFT}_{d,\mathcal{V}}(\mathcal{S}) \simeq \mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}^{\times}).$$

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Theorem (Grady–P., The geometric cobordism hypothesis)

Part I: $\mathfrak{Bord}_d$ is a left adjoint functor:

$$\mathbf{R}\operatorname{Map}(\mathfrak{Bord}_d^{\mathcal{S}}, \mathcal{V}) \simeq \mathbf{R}\operatorname{Map}(\mathcal{S}, \mathcal{V}_d^\times),$$

where $\mathcal{V}_d^\times = \mathsf{FFT}_{d,\mathcal{V}}$, i.e., $\mathcal{V}_d^\times(T \to U) = \mathsf{FFT}_{d,\mathcal{V}}(T \to U)$.

Part II: The evaluation-at-points map

$$\mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) = \mathsf{FFT}_{d,\mathcal{V}}(\mathbf{R}^d \times U \to U) \to \mathcal{V}^\times(U)$$

is a weak equivalence of simplicial sets.

Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the smooth Oka principle (Berwick-Evans–Boavida de Brito–P.).
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (Grady) The Freed–Hopkins conjecture (Conjecture 8.37 in *Reflection positivity and invertible topological phases*)
- Construction of prequantum FFTs from geometric/topological data.
- Quantization of functorial field theories.

Recipe: computing the space of FFTs in practice

Step 1 Compute $\mathcal{V}_d^\times$ (once for every $\mathcal{V}$).

Step 1a Guess a candidate W for $\mathcal{V}_d^\times$. (Standardized guesses exist.)

Step 1b Guess a map $W \to \mathcal{V}_d^\times$. (Typically straightforward.)

Step 1c For every $U \in \mathrm{Cart}$, prove that

$$W(\mathbf{R}^d \times U \to U) \to \mathcal{V}_d^\times(\mathbf{R}^d \times U \to U) \to \mathcal{V}^\times(U)$$

is a weak equivalence. (Easy.)

Step 2 Compute $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, \mathcal{V}_d^\times)$ as $\mathbf{R}\,\mathrm{Map}(\mathcal{S}, W)$. (Like differential cohomology.)

Example: the prequantum Chern–Simons theory (1)

Input data:

- G: a Lie group;
- $\mathcal{S} = \mathsf{B}_\nabla G$ (fiberwise principal G-bundles with connection);
- $\mathcal{V} = \mathsf{B}^3\mathrm{U}(1)$ (a single k-morphism for $k < 3$; 3-morphisms are $\mathrm{U}(1)$ as a Lie group).

Output data: a fully extended 3-dimensional G-gauged FFT:

$$\mathfrak{Bord}_3^{\mathsf{B}_\nabla G} \to \mathsf{B}^3\mathrm{U}(1).$$

- Closed 3-manifold $M \mapsto$ the Chern–Simons action of M;
- Closed 2-manifold $B \mapsto$ the prequantum line bundle of B;
- Closed 1-manifold $C \mapsto$ the Wess–Zumino–Witten gerbe (B-field) of C (Carey–Johnson–Murray–Stevenson–Wang);
- Point $\mapsto$ the Chern–Simons 2-gerbe (Waldorf).

Example: the prequantum Chern–Simons theory (2)

Step 1 Compute $\mathcal{V}_3^\times = (\mathrm{B}^3\mathrm{U}(1))_3^\times$.

Step 1a W is the fiberwise Deligne complex of $T \to U$:

$$W(T \to U) = \Omega^3 \leftarrow \Omega^2 \leftarrow \Omega^1 \leftarrow \mathrm{C}^\infty(T, \mathrm{U}(1)).$$

Step 1b $W \to \mathcal{V}_3^\times$: a fiberwise 3-form ω on $T \to U$
$\mapsto$ framed FFT: 3-bordism $B \mapsto \exp(\int_B \omega)$.

Step 1c The composition

$$W(T \to U) \to \mathcal{V}_3^\times(T \to U) \to \mathcal{V}^\times(U) = \mathrm{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(T, \mathrm{U}(1))$$

is a weak equivalence by the Poincaré lemma.

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is a weak equivalence by the Poincaré lemma.

Step 2 Construct a point in

$$\mathbf{R}\,\mathrm{Map}(\mathrm{B}_\nabla G, W)$$
$$= \mathbf{R}\,\mathrm{Map}(\Omega^1(-, \mathfrak{g})//\mathrm{C}^\infty(-, G), \mathrm{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))).$$

(Brylinski–McLaughlin 1996, Fiorenza–Sati–Schreiber 2013)

Step 2′ Even better: can compute the whole space $\mathbf{R}\,\mathrm{Map}(\mathrm{B}_\nabla G, W)$.

Example: the prequantum Chern–Simons theory (2)

Step 1 Result: $\mathcal{V}_3^\times = (\mathsf{B}^3\mathrm{U}(1))_3^\times = \mathsf{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))$.

Step 2 Construct a point in

$$\begin{aligned} &\mathbf{R}\,\mathrm{Map}(\mathsf{B}_\nabla G, W) \\ &= \mathbf{R}\,\mathrm{Map}(\Omega^1(-,\mathfrak{g})//\mathrm{C}^\infty(-, G), \mathsf{B}^3\mathrm{C}^\infty_{\mathsf{fconst}}(-, \mathrm{U}(1))). \end{aligned}$$

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Quantization of functorial field theories

X: the prequantum geometric structure
Y: the quantum geometric structure (e.g., a point)

$$\begin{array}{ccc} \mathsf{FFT}_{d,\mathcal{V}}(X) & \xrightarrow[\simeq]{\mathsf{GCH}} & \mathbf{R}\,\mathrm{Map}(X, \mathcal{V}_d^\times) \\ \Big\downarrow{\scriptstyle\int} & & \Big\downarrow{\scriptstyle Q} \\ \mathsf{FFT}_{d,\mathcal{V}}(Y) & \xrightarrow[\mathsf{GCH}]{\simeq} & \mathbf{R}\,\mathrm{Map}(Y, \mathcal{V}_d^\times) \end{array}$$

$d = 1$: recover the Spin^c geometric quantization when X is a smooth manifold, $Y = \mathrm{Riem}_{1|1}$, $\mathcal{V} =$ Fredholm complexes.