

# The stringor bundle

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Summary

References

2005: In an unpublished note [ST05] Stephan Stolz and Peter Teichner make a proposal what the *stringor bundle* of a string manifold should be. It is supposed to play the role of the spinor bundle for fermionic strings.

The **stringor bundle** is a Hilbert space bundle  $\mathcal{F}$  over the free loop space  $LM$  of  $M$ , such that the fibre  $\mathcal{F}_\gamma$  over a loop of the form  $\gamma = \beta_1 \cup \beta_2$  is a bimodule

$$\mathcal{A}_{\beta_1} \circlearrowleft \mathcal{F}_\gamma \circlearrowright \mathcal{A}_{\beta_2}$$

for von Neumann algebras  $\mathcal{A}_\beta$  associated to paths  $\beta$ .

Moreover, there is a *Connes fusion product*

$$\mathcal{F}_{\beta_1 \cup \beta_2} \boxtimes_{\mathcal{A}_{\beta_2}} \mathcal{F}_{\beta_2 \cup \beta_3} \cong \mathcal{F}_{\beta_1 \cup \beta_3}.$$

2020: Kristel-KW [KWb, KWc, KWa]: constructed this fusion product rigorously.

Goal of this talk: give a neat & complete construction of the stringor bundle and the setting it lives in, fully analogous to the spinor bundle.

①

Spin group

↓

String 2-group

②

Representation on a  
vector space

↓

Representation on a  
2-vector space

③

Associate to a spin  
structure to obtain a  
vector bundle

↓

Associate to a string  
structure to obtain a  
2-vector bundle

Joint work with Peter Kristel and Matthias Ludewig:  
[KLWa, KLWb, KLWc]

The string 2-group

The stringor representation

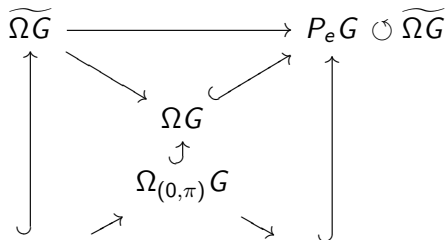
The stringor bundle

Features and applications

2005: Baez, Crans, Schreiber, and Stevenson (BCSS) constructed the **string 2-group** as a crossed module of Fréchet Lie groups [BCSS07].

$$G = \text{Spin}(d)$$

$\widetilde{\Omega G}$  basic central extension



action  
difficult  
to define

2023 [LW]:  $\widetilde{\Omega_{(0,\pi)} G} \longrightarrow P_e G^{flat} \circlearrowright \widetilde{\Omega_{(0,\pi)} G}$

action simple  
& canonical

Indeed:  $\beta \cdot \Phi := \widetilde{\beta \cup \beta} \cdot \Phi \cdot \widetilde{\beta \cup \beta}^{-1}$

Only one issue: **Peiffer identity** – satisfied because  $\widetilde{\Omega G}$  is *disjoint-commutative*.

Our modification of the BCSS string 2-group gives a simple and canonical model as a strict Fréchet Lie 2-group.

It is a **central 2-group extension**:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & BU(1) & \longrightarrow & \text{String}(d) & \longrightarrow & \text{Spin}(d)_{dis} \longrightarrow 1 \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
 U(1) & \longrightarrow & \Omega_{(0,\pi)} \widetilde{\text{Spin}}(d) & \longrightarrow & \text{Spin}(d) & & \\
 \downarrow & & \downarrow & & \parallel & & \\
 * & \longrightarrow & P_e \text{Spin}(d)^{flat} & \xrightarrow{\text{ev}_1} & \text{Spin}(d) & & 
 \end{array}$$

It is classified by its **k-invariant** (Baez-Lauda [BL04]/Schommer-Pries [SP11]):

$$\begin{array}{ccc}
 k_{\text{String}(d)} \in H_{SM}^3(\text{Spin}(d), U(1)) & & \\
 \downarrow & \cong & \\
 1 \in \mathbb{Z} & & H^4(B\text{Spin}(d), \mathbb{Z}) \\
 & & \cong \\
 & & \mathbb{Z}
 \end{array}$$

The string 2-group

**The stringor representation**

The stringor bundle

Features and applications



Finding a useful representation of the string 2-group was an open problem since the invention of the BCSS model in 2005. One important handle is the choice of a model for **2-vector spaces**:

- ▶ Module categories over  $\text{Vect}_k$  (Kapranov-Voevodsky)
- ▶ Categories internal to  $\text{Vect}_k$  (Baez-Crans)
- ▶ Finite abelian linear categories (TQFT context)
- ▶  $\vdots$
- ▶ 2-vector spaces := algebras, bimodules, intertwiners (Schreiber)

vector spaces

2-vector spaces ( = algebras, ...)

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isomorphism

Morita equivalence

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general linear group

$\text{Bimod}(A)^\times$

$\|\!|$  if  $A$  is Picard-surjective [KLWb]

$(A^\times \xrightarrow{c} \text{Aut}(A) \circlearrowleft A^\times) := \text{AUT}(A)$

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representation

2-group homomorphism  $R : \Gamma \rightarrow \text{AUT}(A)$

---

---

inner product space

$C^*$ -algebra

---

unitary group

$(U(A) \xrightarrow{c} \text{Aut}^*(A) \circlearrowleft U(A)) := \text{AUT}^*(A)$

---

unitary representation

2-group morphism  $R : \Gamma \rightarrow \text{AUT}^*(A)$

---

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Hilbert space

von Neumann algebra

Goal: construct a unitary representation

$$\begin{array}{ccc}
 \text{String}(d) & \xrightarrow{R} & \text{AUT}^*(A) \\
 \downarrow \wr & & \downarrow \wr \\
 \widetilde{\Omega_{(0,\pi)}\text{Spin}(d)} & \longrightarrow & \text{U}(A) \\
 \downarrow & & \downarrow \\
 P_e\text{Spin}(d)^{\text{flat}} & \longrightarrow & \text{Aut}^*(A)
 \end{array}$$

Construction of the bottom layer:

- ▶ Real Hilbert space  $V := L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d) \supseteq V_{(0,\pi)}$
- ▶ Clifford  $C^*$ -algebra  $\text{Cl}(V) \supseteq \text{Cl}(V_{(0,\pi)}) =: A$
- ▶ Action:  $\Omega\text{O}(d) \xrightarrow{\text{pointwise}} \text{O}(V) \xrightarrow{\text{Bogoliubov}} \text{Aut}^*(\text{Cl}(V))$

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \uparrow \\
 P_e\text{Spin}(d)^{\text{flat}} & \longrightarrow & P_e\text{O}(d)^{\text{flat}} & \longrightarrow & \text{O}(V_{(0,\pi)}) & \longrightarrow & \text{Aut}^*(A)
 \end{array}$$

Some classical representation theory (Araki [Ara87], Pressley-Segal [PS86], Plymen-Robinson [PR94],...)

- ▶ *APS-Lagrangian* of positive modes of the Dirac operator,

$$L \subseteq V = L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d).$$

- ▶ *Fock space*  $\mathcal{F}_L$  with representation  $\mathbb{C}l(V) \hookrightarrow B(\mathcal{F}_L)$ .

Need this!

$$\begin{array}{ccccccc}
 & & & & \rightarrow & U(A) \subseteq A \subseteq \mathbb{C}l(V) \subseteq B(\mathcal{F}_L) & \\
 & & & \nearrow & \text{---} & & \\
 \widetilde{\Omega}_{(0,\pi)}\text{Spin}(d) \subseteq \widetilde{\Omega}\text{Spin}(d) & \longrightarrow & \text{Imp}(V) & \ni & u & \text{s.t. } \theta_g(a) = uau^* & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \Omega\text{Spin}(d) & \longrightarrow & O_{\text{res}}(V) & \ni & g & & 
 \end{array}$$

Unfortunately, the dashed factorization does not exist,  $A$  is too small!

Solution: we complete to a von Neumann algebra:

$$N := A'' \subseteq \mathbb{C}l(V)'' = B(\mathcal{F}).$$

## Theorem (Kristol-Ludewig-KW 2022 [KLWb])

1. *The desired factorization exists, and we get a commutative diagram*

$$\begin{array}{ccc} \widetilde{\Omega}_{(0,\pi)}\text{Spin}(d) & \longrightarrow & U(N) \\ \downarrow & & \downarrow \\ P_e\text{Spin}(d)^{\text{flat}} & \longrightarrow & \text{Aut}^*(N) \end{array}$$

2. *The diagram is a 2-group homomorphism and thus a unitary representation*

$$\text{String}(d) \rightarrow \text{AUT}^*(N).$$

3. *This representation is continuous when  $\text{Aut}^*(N)$  is equipped with Haagerup's  $u$ -topology.*

The string 2-group

The stringor representation

**The stringor bundle**

Features and applications

2-vector space = algebra

2-vector bundle = algebra bundle ?

- ▶ classified by  $H^0(M, \mathbb{Z}_2) \times \text{Tor}(H^3(M, \mathbb{Z}))$
- ▶ do not glue along bimodule bundles!

Better definition:

2-vector bundle =

a  $\mathcal{A}_\alpha$ - $\mathcal{A}_\beta$ -bimodule bundle

$$\begin{array}{c} \mathcal{A}_\alpha \\ \downarrow \\ U_\alpha \end{array}$$

$$\begin{array}{c} \mathcal{M}_{\alpha\beta} \\ \downarrow \\ U_\alpha \cap U_\beta \end{array}$$

$$\mathcal{M}_{\alpha\beta} \otimes_{\mathcal{A}_\beta} \mathcal{M}_{\beta\gamma} \cong \mathcal{M}_{\alpha\gamma}$$

Some facts [KLWa]:

- ▶ algebra bundles as well as bundle gerbes are 2-vector bundles
- ▶ 2-vector bundles form a 2-stack
- ▶ 2-line bundles classified by  $H^0(M, \mathbb{Z}_2) \times H^3(M, \mathbb{Z})$
- ▶ there is a von Neumann version [KLWc]:  $\otimes_{\mathcal{A}} \rightsquigarrow \boxtimes$

Speculation:

- ▶ classified by a version of elliptic cohomology



Suppose  $N$  is a von Neumann algebra and we have a principal  $\text{AUT}^*(N)$ -2-bundle over  $M$ :

$$\begin{array}{ccc}
 \mathcal{P}_{\alpha\beta} & \xrightarrow{\phi} & \text{Aut}^*(N) \\
 \downarrow \text{U}(N) & & \\
 U_\alpha \cap U_\beta & & 
 \end{array}
 \qquad
 \mathcal{P}_{\alpha\beta} \otimes \mathcal{P}_{\beta\gamma} \cong \mathcal{P}_{\alpha\gamma}$$

$\downarrow$  Lemma

The **associated 2-Hilbert space bundle** is the following:

$$\begin{array}{ccc}
 \mathcal{A}_\alpha := U_\alpha \times N & \mathcal{M}_{\alpha\beta} := \phi(\mathcal{P}_{\alpha\beta} \times_{\text{U}(N)} N) & \mathcal{M}_{\alpha\beta} \boxtimes \mathcal{M}_{\beta\gamma} \cong \mathcal{M}_{\alpha\gamma} \\
 \downarrow & \downarrow & \\
 U_\alpha & U_\alpha \cap U_\beta & 
 \end{array}$$

If  $\mathcal{P}$  is a principal 2-bundle for a general crossed module  $\Gamma$ , and  $R : \Gamma \rightarrow \text{AUT}^*(N)$  is a continuous representation, then we push along  $R$  and form the associated 2-Hilbert space bundle for  $R_*(\mathcal{P})$ .

①

String 2-group

②

Representation on the  
2-vector space  $N = A''$

③

2-vector bundle  
associated to a string  
structure

$$\mathcal{S}(M) \times_{\text{String}(d)} N$$

!!

Stringor bundle

The string 2-group

The stringor representation

The stringor bundle

**Features and applications**

(A) **Transgression** to loop space:

higher structure over $M$	$\rightsquigarrow$	ordinary structure over $LM$ compatible with loop fusion and thin homotopies
bundle gerbe over $M$	$\rightsquigarrow$	line bundle over $LM$ with fusion product and thin homotopy equivariant structure [Wal16, Wal12]
stringor bundle	$\rightsquigarrow$	Stolz-Teichner's loop space bundle with its Connes fusion product

(B) Fermionic string theory as a **smooth functorial field theory**:

Proposal: the stringor bundle is its value “on the point”.

⇒ its transgression, i.e., Stolz-Teichner’s loop space bundle, becomes the value “on the circle”.

⇒ the Connes fusion product is what the theory assigns to a pair of pants.

Conjecture: the value on the torus is the fermionic action functional (under Bunke’s anomaly cancellation mechanism [Bun11] using the given string structure).

- (C) There is a notion of a twisted 2-vector bundle, and a construction of a **twisted stringor bundle** for non-string manifolds, analogous to Mathai-Melrose's work on twisted spinor bundles and fractional indices.
- (D) In ongoing work we try to use a string connection in order to define a **covariant derivative** on the stringor bundle.

The main open questions one could then attack are:

- ▶ Can one define a “Dirac” operator on sections of the stringor bundle?
- ▶ What is the index of this operator, and does it compute the Witten genus?
- ▶ What can it say about the existence of metrics with positive Ricci curvature on  $M$  ?

## Summary: the stringor bundle

- ▶ it is a von Neumann 2-vector bundle over a string manifold
- ▶ it can be defined as an associated bundle,

$$\mathcal{S}(M) \times_{\text{String}(d)} N,$$

using a representation of the string 2-group

- ▶ it is the value of a smooth functorial field theory on the point, modelling fermionic strings

**Thank you!**

## References

- [Ara87] Huzihiro Araki, “Bogoliubov automorphisms and Fock representations of canonical anticommutation relations”. *Contemp. Math.*, 62:23–141, 1987.
- [BCSS07] John C. Baez, Alissa S. Crans, Danny Stevenson, and Urs Schreiber, “From loop groups to 2-groups”. *Homology, Homotopy Appl.*, 9(2):101–135, 2007. [[arxiv:math.QA/0504123](#)].
- [BL04] John C. Baez and Aaron D. Lauda, “Higher-dimensional algebra V: 2-groups”. *Theory Appl. Categ.*, 12:423–491, 2004. [[arxiv:math/0307200](#)].
- [Bun11] Ulrich Bunke, “String Structures and Trivialisations of a Pfaffian Line Bundle”. *Commun. Math. Phys.*, 307(3):675–712, 2011. [[arxiv:0909.0846](#)].
- [KLWa] Peter Kristel, Matthias Ludewig, and Konrad Waldorf, “2-vector bundles”. Preprint. [[arxiv:2106.12198](#)].



- [KLWb] Peter Kristel, Matthias Ludewig, and Konrad Waldorf, "The insidious bicategory of algebra bundles". Preprint. [\[arxiv:2204.03900\]](#).
- [KLWc] Peter Kristel, Matthias Ludewig, and Konrad Waldorf, "A representation of the string 2-group". Preprint. [\[arxiv:2206.09797\]](#).
- [KWa] Peter Kristel and Konrad Waldorf, "Connes fusion of spinors on loop space". Preprint. [\[arxiv:2012.08142\]](#).
- [KWb] Peter Kristel and Konrad Waldorf, "Fusion of implementers for spinors on the circle". *Adv. Math.*, to appear. [\[arxiv:1905.00222\]](#).
- [KWc] Peter Kristel and Konrad Waldorf, "Smooth Fock bundles, and spinor bundles on loop space". *J. Differential Geom.*, to appear. [\[arxiv:2009.00333\]](#).
- [LW] Matthias Ludewig and Konrad Waldorf, "Loop group extensions and Lie 2-groups". In preparation.
- [PR94] Roger Plymen and Paul Robinson, *Spinors in Hilbert space*. Cambridge Univ. Press, 1994.

- [PS86] A. Pressley and Graeme Segal, *Loop groups*. Oxford Univ. Press, 1986.
- [SP11] Chris Schommer-Pries, “Central extensions of smooth 2-groups and a finite-dimensional string 2-group”. *Geom. Topol.*, 15:609–676, 2011. [[arxiv:0911.2483](#)].
- [ST05] Stephan Stolz and Peter Teichner, “The spinor bundle on loop spaces”. Unpublished, 2005. .
- [Wal12] Konrad Waldorf, “Transgression to loop spaces and its inverse, III: Gerbes and thin fusion bundles”. *Adv. Math.*, 231:3445–3472, 2012. [[arxiv:1109.0480](#)].
- [Wal16] Konrad Waldorf, “Transgression to loop spaces and its inverse, II: Gerbes and fusion bundles with connection”. *Asian J. Math.*, 20(1):59–116, 2016. [[arxiv:1004.0031](#)].