

# Quasi-Alternating Links, Examples and Obstructions

**Nafaa Chbili**

**United Arab Emirates University**

Quantum Field Theories and Cobordisms

NYUAD

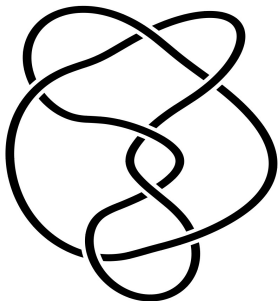
March 15-18, 2023

# Outline

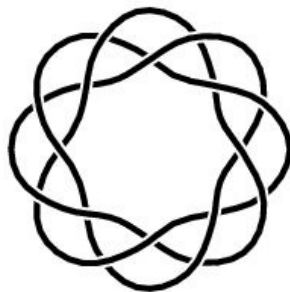
- 1 Alternating links
- 2 QA links, Examples and Properties
- 3 Invariants of QA links

# Alternating links

- A link diagram is alternating if the overpass and the underpass alternate as one travels along the diagram.
- A link is said to be alternating if it can be represented by an alternating diagram. Otherwise, the link is non-alternating.

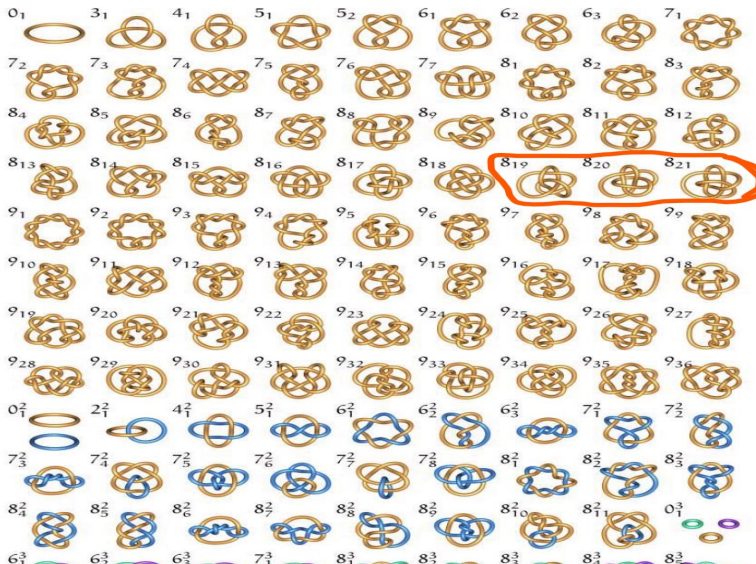


Alternating knot diagram



Non-alternating knot diagram

# Alternating links



# Alternating links

Suppose that  $K$  is an alternating knot,  $\Delta_K(t)$  its Alexander polynomial and  $V_K(t)$  its Jones polynomial.

- 1 The genus of the knot  $g(K)$  is the degree of its "symmetric" Alexander polynomial.
- 2 The coefficients of  $\Delta_K(t)$  alternate in sign and have no gaps.
- 3 The span of  $V_K(t)$  is equal to the crossing number of  $K$ .
- 4 The coefficients of  $V_K(t)$  alternate in sign and have no gaps (if  $K$  is prime and not a torus knot).

# Alternating links

Given an alternating link  $L$  and  $\Sigma_L$  the branched double-cover of  $L$ .

- ①  $\Sigma_L$  is an  $L$ -space (i.e. Heegard Floer Homology of  $\Sigma_L$  is determined by  $\det(L)$ ), [Ozsváth and Szabó];
- ②  $\Sigma_L$  bounds a negative definite 4-manifold  $W$  with  $H_1(W) = 0$ , [Ozsváth and Szabó];
- ③ The reduced ordinary Khovanov homology group of  $L$  is thin,  $KH^{i,j}(L)$  is trivial whenever  $i - j \neq \frac{\sigma(L)}{2}$ , [Lee];
- ④ The  $\mathbb{Z}_2$  link Floer homology group of  $L$  is thin, [Rasmussen, Ozsváth- Szabó];
- ⑤ The reduced odd Khovanov homology group of  $L$  is thin, [Ozsváth, Rasmussen and Szabó];

# Topological Characterization of Alternating links

A topological characterization of alternating link is obtained by Greene and Howie.

## Greene, Howie

A link is alternating if and only if it simultaneously bounds both positive and negative definite spanning surfaces.

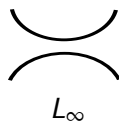
# Quasi-alternating links

## Definition

The set  $\mathcal{Q}$  of quasi-alternating links is the smallest set such that:

- The unknot belongs to  $\mathcal{Q}$ .
- If  $L$  is a link with a diagram  $D$  having a crossing  $c$  such that
  - ① Both smoothing of  $D$  at  $c$ ,  $L_0$  and  $L_\infty$  are in  $\mathcal{Q}$ ,
  - ②  $\det(L_0), \det(L_\infty) \geq 1$ ,
  - ③  $\det(L) = \det(L_0) + \det(L_\infty)$ ; then  $L \in \mathcal{Q}$

We say that  $L$  is quasi-alternating at the crossing  $c$  with quasi-alternating diagram  $D$ .



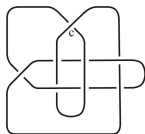


# Quasi-alternating links

- 1 Any non-split alternating link is QA at any crossing of any reduced alternating diagram [Ozsváth and Szabó].
- 2 First examples of non-alternating QA knots are  $8_{20}$  and  $8_{21}$ .
- 3 QA diagrams of knots with  $\leq 9$  crossings are given by Manolescu.



$8_{20}$



$8_{21}$

## From the definition

- The determinant of a QA link is positive and it is equal to 1 if and only if  $L$  is the unknot;
- If  $K_1$  and  $K_2$  are two QA knots, then so is their connected sum  $K_1 \sharp K_2$ , [Champanerkar and Kofman];
- If  $L$  is quasi-alternating, then so is its mirror image  $L!$ ;

### Problem

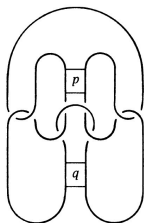
Decide whether a given knot or link is QA or not.

## Obstruction for QA links

- ① The branched double-cover of a QA link  $L$ ,  $\Sigma_L$  is an  $L$ -space (i.e. Heegard Floer Homology of  $\Sigma_L$  is determined by  $\det(L)$ ) [Ozsváth and Szabó];
- ② The branched double-cover of a QA link bounds a negative definite 4-manifold  $W$  with  $H_1(W) = 0$ , [Ozsváth and Szabó];
- ③ The reduced ordinary Khovanov homology group of a QA link is thin;  $KH^{i,j}(L)$  is trivial whenever  $i - j \neq \frac{\sigma(L)}{2}$ ; [Manolescu and Ozsváth];
- ④ The  $\mathbb{Z}_2$  link Floer homology group of a QA link is thin [Manolescu and Ozsváth];
- ⑤ The reduced odd Khovanov homology group of a QA link is thin, [Ozsváth, Rasmussen and Szabó];

## Homologically thin non QA links

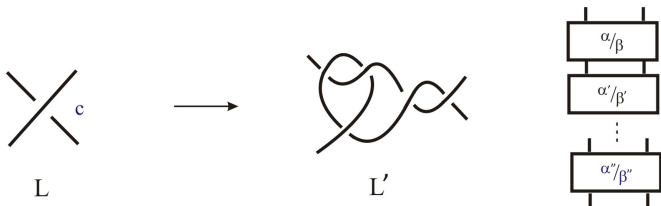
- Torus links  $T(p, q)$ ,  $p, q \geq 3$  are Khovanov homologically thick so not QA.
- The first homologically thin non QA knot:  $11n_{50}$ ; [Greene]
- An infinite family of homologically thin, hyperbolic non QA knots: Kanenobu knots of type  $K(-10n, 10n + 3)$  [Greene-Watson].



The  $K(p, q)$  Kanenobu knot.

## Examples of QA links

Champanerkar-Kofman: Given a link  $L$  with QA diagram  $D$  at a crossing  $c$ . Then any link diagram obtained from  $D$  by replacing  $c$  by an alternating rational tangle of the same type is QA.



$$\frac{8}{3} = 2 + \frac{1}{1 + \frac{1}{2}}$$

This twisting technique has been applied to classify all QA Montesinos links.

## QA Montesinos Links

This theorem was first conjectured by Qazaqzeh-Qublan-C and Champanerkar-Ording, then recently proved by Issa.

### Theorem

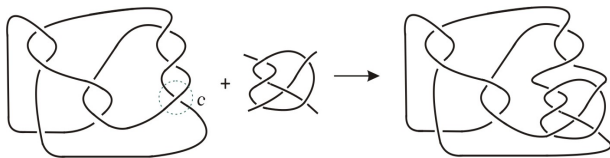
Let  $L = M(e; t_1, \dots, t_p)$  be a Montesinos link in standard form, that is, where  $t_i = \frac{\alpha_i}{\beta_i} > 1$  and  $\alpha_i, \beta_i > 0$  are coprime for all  $i$ . Then  $L$  is quasi-alternating if and only if

- (1)  $e < 1$ , or
- (2)  $e = 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ , or
- (3)  $e > p - 1$ , or
- (4)  $e = p - 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ .

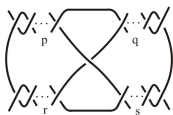
## More Examples

### Kaur-C

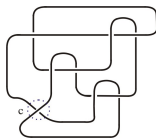
Let  $L$  be a QA link diagram and  $c$  be a QA crossing. If  $L'$  is a link obtained from  $L$  by replacing crossing  $c$  by an alternating tangle  $T$  (algebraic or non-algebraic) of same type, then  $L'$  is QA at every crossing of  $T$  in  $L'$ .



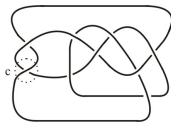
## More Examples of non-alternating QA knots



$T_{pqrs}$



$8_{21}$



$9_{47}$

$K(T)$ : Diagram obtained by replacing crossing  $c$  in knot diagram  $K$  by a tangle  $T$

$\widehat{T}$ : Clock-wise  $90^\circ$  rotation of  $T$  followed by switching crossings.

$13n329 : 8_{21}(\widehat{T}_{1112})$	$13n357 : 8_{21}(T_{2111})$
$13n351 : 8_{21}(\widehat{T}_{2111})$	$13n520 : 8_{21}(T_{1121})$
$13n513 : 8_{21}(\widehat{T}_{1121})$	$13n530 : 8_{21}(\widehat{T}_{1211})$
$13n525 : 8_{21}(\widehat{T}_{1211})$	$14n20087 : 9_{47}(T_{1211})$
$13n344 : 8_{21}(T_{1112})$	$14n20090 : 9_{47}(T_{1121})$



In addition to the classification of QA pretzel and Montesinos links,

- QA links with braid index 3 are completely characterized; [Greene]
- QA links with determinants  $\leq 7$  have been characterized; [Lidman-Sivek]
- QA links with at most 11 crossings have been determined;

# The Jones polynomial

The Jones polynomial is an isotopy invariant of oriented links defined by:

$$V_{\bigcirc}(t) = 1$$

$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)V_{L_0}(t),$$

where  $L_+$ ,  $L_-$  and  $L_0$  are 3 links as pictured below:



We can always write the Jones polynomial of any link  $L$  as follows:

$$V_L(t) = t^r \sum_{i=0}^m a_i t^i, \text{ where } m \geq 0, a_0 \neq 0 \text{ and } a_m \neq 0.$$

$$m := \text{span}(V_L)$$

Let  $L$  be a QA link.

- The coefficients of  $V_L(t)$  alternate in sign. This is a consequence of its thin Khovanov homology.
- $KH^{i,j}(L)$  has no gaps of length larger than 1 in the differential grading, thus  $V_L(t)$  has no gap of length larger than one [Qazaqzeh-C].

$j,i$	-3	-2	-1	0
-1				1
-3				1
-5		1		
-7				
-9	1			

Table: The Khovanov homology of the trefoil knot.

## Colored Jones Polynomials

Let  $J_{N,L}(q)$  be the Colored Jones polynomial of  $L$ .

If  $L$  is alternating  $J_{N,L}(q)$  satisfies the Head-tail property:

The first and the last  $k$  coefficients of the polynomial do not depend on  $N$  as long as  $N > k$ . For example, for  $J_{N,6_3}(q)$  we have:

$$N = 2: \quad 1 - 2q + 2q^2 - 2q^3 + 2q^4 - q^5 + q^6$$

$$N = 3: \quad 1 - 2q + 4q^3 - 5q^4 + 6q^6 + \dots - q^{14} + 3q^{15} - q^{16} - q^{17} + q^{18}$$

$$N = 4: \quad 1 - 2q + 2q^3 + q^4 - 4q^5 - 2q^6 + \dots - 2q^{29} - 3q^{30} + 3q^{32} - q^{34} - q^{35} + q^{36}$$

$$N = 5: \quad 1 - 2q + 2q^3 - q^4 + 2q^5 - 6q^6 + \dots - 2q^{53} - q^{54} + 4q^{55} - q^{58} - q^{59} + q^{60}$$

$$N = 6: \quad 1 - 2q + 2q^3 - q^4 - 2q^7 + q^8 + \dots - 3q^{82} + 3q^{84} + q^{85} - q^{88} - q^{89} + q^{90}$$

$$N = 7: \quad 1 - 2q + 2q^3 - q^4 - 2q^6 + 4q^7 + \dots + 4q^{119} + q^{121} - q^{124} - q^{125} + q^{126}$$

**Question.** Head-tail property is not true QA links. Does  $J_{N,L}(q)$  satisfy any similar property?

# Alexander Polynomial

Recall that if  $K$  is an alternating knot and  $\Delta_K(t) = \sum_{-m}^m \alpha_i t^i$ , then:

- ①  $g(K) = m$ .
- ②  $\alpha_i \alpha_{i+1} < 0$ , for all  $-m \leq i \leq m-1$ .

Condition 1 is true for QA knots and knots which are thin in Floer Homology in general, [Ozsváth-Szabó].

**Question.** Does condition 2 extend to quasi-alternating links?

**Remark.** Fox Trapezoid Conjecture does not hold for quasi-alternating links.

# Open Problem

**Problem.** Give a topological characterization of QA links.