Quantum field theories on Lorentzian manifolds

Alexander Schenkel

School of Mathematical Sciences, University of Nottingham

University of Nottingham UK | CHINA | MALAYSIA

THE ROYAL SOCIETY

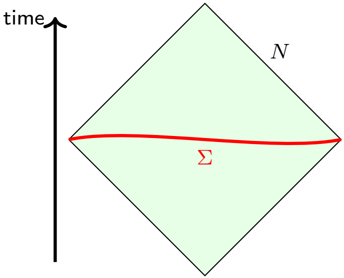
Geometric/Topological Quantum Field Theories and Cobordisms,
15–18 March 2023, NYU Abu Dhabi.

Based on a research program with Marco Benini, with contributions from

S. Bruinsma, S. Bunk, V. Carmona, C. Fewster, L. Giorgetti, A. Grant-Stuart, J. MacManus, G. Musante, M. Perin, J. Pridham, P. Safronov, U. Schreiber, R. Szabo and L. Woike.

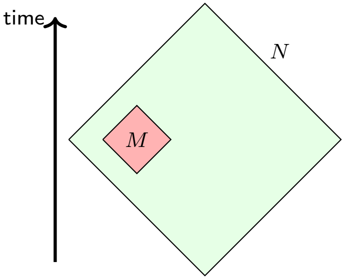
Some background on Lorentzian spacetimes

- **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N



Some background on Lorentzian spacetimes

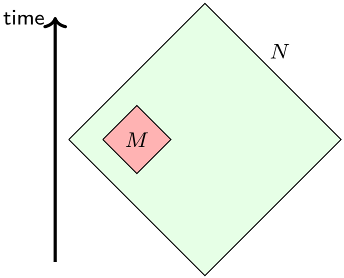
- **Spacetime** $:=$ oriented and time-oriented globally hyperbolic Lorentzian manifold N
- **Spacetime embedding** $:=$ orientation and time-orientation preserving isometric embedding $f : M \to N$ s.t. $f(M) \subseteq N$ is open and causally convex



Some background on Lorentzian spacetimes

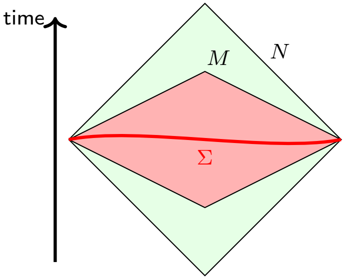
- **Spacetime** $:=$ oriented and time-oriented globally hyperbolic Lorentzian manifold N
- **Spacetime embedding** $:=$ orientation and time-orientation preserving isometric embedding $f : M \to N$ s.t. $f(M) \subseteq N$ is open and causally convex

Def: Denote by $\mathbf{Loc}_m$ the category of m-dim. spacetimes and spacetime embeddings.



Some background on Lorentzian spacetimes

- **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N
- **Spacetime embedding** := orientation and time-orientation preserving isometric embedding $f : M \to N$ s.t. $f(M) \subseteq N$ is open and causally convex

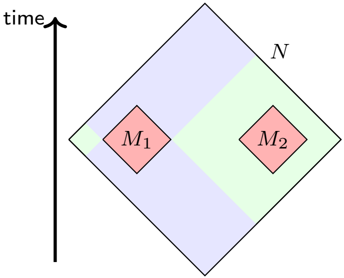


Def: Denote by $\mathbf{Loc}_m$ the category of m-dim. spacetimes and spacetime embeddings.

- The following (tuples of) $\mathbf{Loc}_m$-morphisms will be important:
  - (i) Cauchy morphism: $f : M \to N$ s.t. $f(M) \subseteq N$ contains Cauchy surface of N

Some background on Lorentzian spacetimes

- **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N
- **Spacetime embedding** := orientation and time-orientation preserving isometric embedding $f : M \to N$ s.t. $f(M) \subseteq N$ is open and causally convex

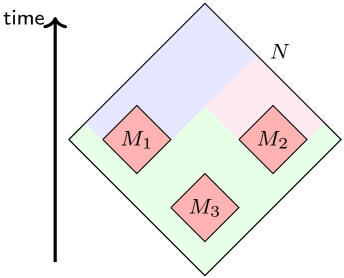


Def: Denote by $\mathbf{Loc}_m$ the category of m-dim. spacetimes and spacetime embeddings.

- The following (tuples of) $\mathbf{Loc}_m$-morphisms will be important:
  - (i) Cauchy morphism: $f : M \to N$ s.t. $f(M) \subseteq N$ contains Cauchy surface of N
  - (ii) Causally disjoint pair: $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$ s.t. $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$

Some background on Lorentzian spacetimes

- **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N
- **Spacetime embedding** := orientation and time-orientation preserving isometric embedding $f : M \to N$ s.t. $f(M) \subseteq N$ is open and causally convex

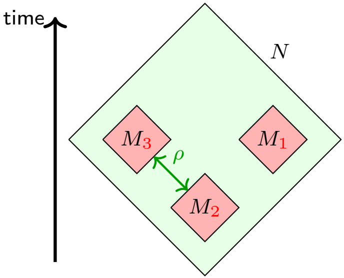


Def: Denote by $\mathbf{Loc}_m$ the category of m-dim. spacetimes and spacetime embeddings.

- The following (tuples of) $\mathbf{Loc}_m$-morphisms will be important:
  - (i) Cauchy morphism: $f : M \to N$ s.t. $f(M) \subseteq N$ contains Cauchy surface of N
  - (ii) Causally disjoint pair: $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$ s.t. $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$
  - (iii) Time-ordered tuple: $\underline{f} = (f_1, \dots, f_n) : \underline{M} = (M_1, \dots, M_n) \to N$ s.t. $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$, for all $i < j$

Some background on Lorentzian spacetimes

- **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N
- **Spacetime embedding** := orientation and time-orientation preserving isometric embedding $f : M \to N$ s.t. $f(M) \subseteq N$ is open and causally convex



Def: Denote by $\mathbf{Loc}_m$ the category of m-dim. spacetimes and spacetime embeddings.

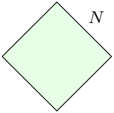
- The following (tuples of) $\mathbf{Loc}_m$-morphisms will be important:
  (i) Cauchy morphism: $f : M \to N$ s.t. $f(M) \subseteq N$ contains Cauchy surface of N
  (ii) Causally disjoint pair: $(f_1 : M_1 \to N) \perp (f_2 : M_2 \to N)$ s.t. $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$
  (iii) Time-ordered tuple: $\underline{f} = (f_1, \dots, f_n) : \underline{M} = (M_1, \dots, M_n) \to N$ s.t. $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$, for all $i < j$
  (iv) Time-orderable tuple: $\underline{f} : \underline{M} \to N$ s.t. there exists $\rho \in \Sigma_n$ (time-ordering permutation) with $\underline{f}\rho = (f_{\rho(1)}, \dots, f_{\rho(n)}) : \underline{M}\rho \to N$ time-ordered

What's a QFT on Lorentzian spacetimes?

- ◇ Inspired by algebraic QFT [Haag/Kastler, Brunetti/Fredenhagen/Verch, ...], one should study the following algebraic structure (for $\mathbf{T}$ = SM (∞-)category)

What's a QFT on Lorentzian spacetimes?

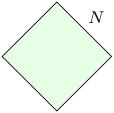
- Inspired by algebraic QFT [Haag/Kastler, Brunetti/Fredenhagen/Verch, ...], one should study the following algebraic structure (for $\mathbf{T} =$ SM (∞-)category)



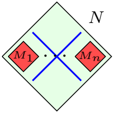
$$\xrightarrow{\text{QFT}} \quad \underbrace{\mathfrak{A}(N) \in \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{T})}_{\text{quantum observables in } N}$$

What's a QFT on Lorentzian spacetimes?

◇ Inspired by algebraic QFT [Haag/Kastler, Brunetti/Fredenhagen/Verch, ...], one should study the following algebraic structure (for $\mathbf{T}$ = SM (∞-)category)



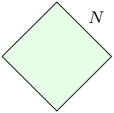
$\overset{\text{QFT}}{\longmapsto}$ $\underbrace{\mathfrak{A}(N) \in \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{T})}_{\text{quantum observables in } N}$



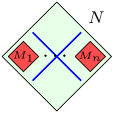
$\longmapsto$ $\underbrace{\bigotimes_{i=1}^{n} \mathfrak{A}(M_i) \longrightarrow \mathfrak{A}(N)}_{\text{composition of causally independent subsystems}}$

What's a QFT on Lorentzian spacetimes?

◇ Inspired by algebraic QFT [Haag/Kastler, Brunetti/Fredenhagen/Verch, ...], one should study the following algebraic structure (for $\mathbf{T}$ = SM (∞-)category)

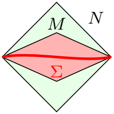


$$\overset{\text{QFT}}{\longmapsto} \quad \underbrace{\mathfrak{A}(N) \in \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{T})}_{\text{quantum observables in } N}$$



$$\longmapsto \quad \underbrace{\bigotimes_{i=1}^{n} \mathfrak{A}(M_i) \longrightarrow \mathfrak{A}(N)}_{\text{composition of causally independent subsystems}}$$

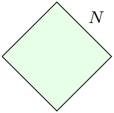
that satisfies the *(homotopy) time-slice axiom*



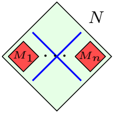
$$\longmapsto \quad \underbrace{\mathfrak{A}(M) \xrightarrow{\sim} \mathfrak{A}(N)}_{\text{defines a concept of time evolution}}$$

What's a QFT on Lorentzian spacetimes?

◇ Inspired by algebraic QFT [Haag/Kastler, Brunetti/Fredenhagen/Verch, . . .], one should study the following algebraic structure (for $\mathbf{T}$ = SM (∞-)category)

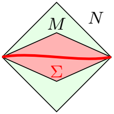


$$\xmapsto{\text{QFT}} \quad \underbrace{\mathfrak{A}(N) \in \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{T})}_{\text{quantum observables in } N}$$



$$\longmapsto \quad \underbrace{\bigotimes_{i=1}^{n} \mathfrak{A}(M_i) \longrightarrow \mathfrak{A}(N)}_{\text{composition of causally independent subsystems}}$$

that satisfies the *(homotopy) time-slice axiom*



$$\longmapsto \quad \underbrace{\mathfrak{A}(M) \xrightarrow{\sim} \mathfrak{A}(N)}_{\text{defines a concept of time evolution}}$$

◇ This is governed by the AQFT operad [Benini/AS/Woike, Benini/Carmona/AS]

$$\mathcal{O}_{(\mathbf{Loc}_m,\perp)}[\mathrm{Cauchy}^{-1}]^{\infty} \simeq \big(\mathcal{P}_{(\mathbf{Loc}_m,\perp)} \otimes \mathsf{uAs}\big)[\mathrm{Cauchy}^{-1}]^{\infty}$$

Classification in low dimensions (for target $\mathbf{T}$ = SM 1-category)

Prop: [Benini/Woike/AS] Given orthogonal category $(\mathbf{C},\perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \,\simeq\, \mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))} \quad ,$$

where $L:\mathbf{C}\to\mathbf{C}[W^{-1}]$ is localization of underlying category.

Classification in low dimensions (for target $\mathbf{T}$ = SM 1-category)

Prop: [Benini/Woike/AS] Given orthogonal category $(\mathbf{C}, \perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \simeq \mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))} \quad ,$$

where $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is localization of underlying category.

- ◇ In low dimensions, these localizations can be determined explicitly. E.g.

Classification in low dimensions (for target $\mathbf{T}$ = SM 1-category)

Prop: [Benini/Woike/AS] Given orthogonal category $(\mathbf{C}, \perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \simeq \mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))} \quad ,$$

where $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is localization of underlying category.

- In low dimensions, these localizations can be determined explicitly. E.g.
  - (i) 1-dim. AQFT $\mathfrak{A}$ on $\mathbf{Loc}_1$ $\Longleftrightarrow$ [Benini/Carmona/AS]

$$\mathbb{R} \circlearrowright \mathfrak{A}(|) \quad = \quad \text{dynamical system with time evolution}$$

$$= \quad \text{quantum mechanics}$$

Classification in low dimensions (for target $\mathbf{T}$ = SM 1-category)

Prop: [Benini/Woike/AS] Given orthogonal category $(\mathbf{C}, \perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \simeq \mathcal{O}_{(\mathbf{C}[W^{-1}], L_*(\perp))} \quad ,$$

where $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is localization of underlying category.

◇ In low dimensions, these localizations can be determined explicitly. E.g.

(i) 1-dim. AQFT $\mathfrak{A}$ on $\mathbf{Loc}_1 \iff$ [Benini/Carmona/AS]

$$\mathbb{R} \circlearrowright \mathfrak{A}(|) \quad = \quad \text{dynamical system with time evolution}$$
$$= \quad \text{quantum mechanics}$$

(ii) 2-dim. *conformal* AQFT $\mathfrak{A}$ on $\mathbf{CLoc}_2 \iff$ [Benini/Giorgetti/AS]

$$\mathrm{Emb}(\mathbb{R})^2 \circlearrowright \mathfrak{A}(\Diamond) \xrightarrow{\ \mathrm{Emb}(\Diamond, \square)\ } \mathfrak{A}(\square) \circlearrowleft \mathrm{Diff}(\mathbb{S}^1)^2$$

Prop: [Benini/Woike/AS] Given orthogonal category $(\mathbf{C}, \perp)$ and $W \subseteq \mathrm{Mor}\,\mathbf{C}$, then

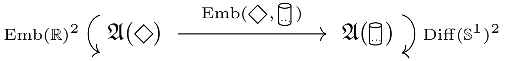
$$\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}] \simeq \mathcal{O}_{(\mathbf{C}[W^{-1}], L_*(\perp))} \quad ,$$

where $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is localization of underlying category.

- In low dimensions, these localizations can be determined explicitly. E.g.
  - (i) 1-dim. AQFT $\mathfrak{A}$ on $\mathbf{Loc}_1 \iff$ [Benini/Carmona/AS]

$$\mathbb{R} \circlearrowright \mathfrak{A}(|) \quad = \quad \text{dynamical system with time evolution}$$
$$= \quad \text{quantum mechanics}$$

  - (ii) 2-dim. *conformal* AQFT $\mathfrak{A}$ on $\mathbf{CLoc}_2 \iff$ [Benini/Giorgetti/AS]



- *Open problem:* Higher dimensions? Some speculations later...

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\operatorname{char} \mathbb{K} = 0$)

- There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:
  - (i) Strict time-slice axiom (projective model structure)

$$\mathbf{AQFT}(\mathbf{C}, \perp)^{W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C}[W^{-1}], L_*(\perp))}}(\mathbf{Ch}_{\mathbb{K}})$$

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\mathrm{char}\,\mathbb{K} = 0$)

- There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:
  - (i) Strict time-slice axiom (projective model structure)

$$\mathbf{AQFT}(\mathbf{C},\perp)^{W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))}}(\mathbf{Ch}_{\mathbb{K}})$$

  - (ii) Homotopy time-slice axiom (left Bousfield localization à la [Carmona])

$$\mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}]^{\infty}}(\mathbf{Ch}_{\mathbb{K}}) \simeq \mathcal{L}_{\widehat{W}}\mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}}(\mathbf{Ch}_{\mathbb{K}})$$

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\mathrm{char}\,\mathbb{K} = 0$)

◇ There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:

(i) Strict time-slice axiom (projective model structure)

$$\mathbf{AQFT}(\mathbf{C},\perp)^{W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))}}(\mathbf{Ch}_{\mathbb{K}})$$

(ii) Homotopy time-slice axiom (left Bousfield localization à la [Carmona])

$$\mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}]^{\infty}}(\mathbf{Ch}_{\mathbb{K}}) \simeq \mathcal{L}_{\widehat{W}}\mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}}(\mathbf{Ch}_{\mathbb{K}})$$

Thm: [Benini/Carmona/AS] The localization functor $L : (\mathbf{C},\perp) \to (\mathbf{C}[W^{-1}], L_*(\perp))$ defines a Quillen adjunction

$$L_! : \mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} \rightleftarrows \mathbf{AQFT}(\mathbf{C},\perp)^{W} : L^* \quad .$$

If L is a *reflective orthogonal localization*, then this is a Quillen equivalence.

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\mathrm{char}\,\mathbb{K} = 0$)

- There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:
  (i) Strict time-slice axiom (projective model structure)

$$\mathbf{AQFT}(\mathbf{C},\perp)^{W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))}}(\mathbf{Ch}_{\mathbb{K}})$$

  (ii) Homotopy time-slice axiom (left Bousfield localization à la [Carmona])

$$\mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} := \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}]_\infty}(\mathbf{Ch}_{\mathbb{K}}) \simeq \mathcal{L}_{\widehat{W}}\mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}}(\mathbf{Ch}_{\mathbb{K}})$$

Thm: [Benini/Carmona/AS] The localization functor $L : (\mathbf{C},\perp) \to (\mathbf{C}[W^{-1}], L_*(\perp))$ defines a Quillen adjunction

$$L_! \,:\, \mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} \rightleftarrows \mathbf{AQFT}(\mathbf{C},\perp)^{W} \,:\, L^* \quad .$$

If L is a *reflective orthogonal localization*, then this is a Quillen equivalence.

!!! Strictification theorems for the homotopy time-slice axiom for AQFTs on $\mathbf{Loc}_1$, $\mathbf{CLoc}_2$ and Haag-Kastler-type $\mathbf{Loc}_m/M$.

Strictifying the time-slice axiom (for $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ with $\operatorname{char}\mathbb{K} = 0$)

◇ There are two (i.g. different) model categories for $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs:

(i) Strict time-slice axiom (projective model structure)

$$\mathbf{AQFT}(\mathbf{C},\perp)^W \;:=\; \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C}[W^{-1}],L_*(\perp))}}(\mathbf{Ch}_{\mathbb{K}})$$

(ii) Homotopy time-slice axiom (left Bousfield localization à la [Carmona])

$$\mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} \;:=\; \mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}[W^{-1}]^\infty}(\mathbf{Ch}_{\mathbb{K}}) \simeq \mathcal{L}_{\widehat{W}}\mathbf{Alg}_{\mathcal{O}_{(\mathbf{C},\perp)}}(\mathbf{Ch}_{\mathbb{K}})$$

Thm: [Benini/Carmona/AS] The localization functor $L : (\mathbf{C},\perp) \to (\mathbf{C}[W^{-1}], L_*(\perp))$ defines a Quillen adjunction

$$L_! \,:\, \mathbf{AQFT}(\mathbf{C},\perp)^{\mathrm{ho}W} \rightleftarrows \mathbf{AQFT}(\mathbf{C},\perp)^W \,:\, L^* \quad .$$

If L is a *reflective orthogonal localization*, then this is a Quillen equivalence.

!!! Strictification theorems for the homotopy time-slice axiom for AQFTs on $\mathbf{Loc}_1$, $\mathbf{CLoc}_2$ and Haag-Kastler-type $\mathbf{Loc}_m/M$.

Rem: Very different behavior to topological QFTs (via locally constant factorization algebras on $\mathbb{R}^m$) $\rightsquigarrow$ $\mathbb{E}_m$-algebras [Lurie, Ayala/Francis]

Construction of free (non-interacting) QFTs on $\mathbf{Loc}_m$

◇ *Input data:* A natural collection $\{\mathcal{F}(M), Q_M, \omega_M\}_{M\in\mathbf{Loc}_m}$ of free BV theories [Costello/Gwilliam], i.e. $(\mathcal{F}(M), Q_M)$ is a complex of differential operators and ω_M is a (-1)-shifted symplectic structure.

Construction of free (non-interacting) QFTs on $\mathbf{Loc}_m$

- *Input data:* A natural collection $\{\mathcal{F}(M), Q_M, \omega_M\}_{M\in\mathbf{Loc}_m}$ of free BV theories [Costello/Gwilliam], i.e. $(\mathcal{F}(M), Q_M)$ is a complex of differential operators and ω_M is a (-1)-shifted symplectic structure.

- *Central hypothesis:* Green-hyperbolic complexes, i.e. there exists (pseudo-)natural family of *retarded/advanced Green's homotopies*

$$\Big\{\Lambda_K^\pm \in \big[\mathcal{F}_K(M), \mathcal{F}_{J_M^\pm(K)}(M)\big]^{-1} \,:\, \partial\Lambda_K^\pm = \mathrm{incl}\Big\}_{K\subseteq M \text{ compact}}$$

Construction of free (non-interacting) QFTs on $\mathbf{Loc}_m$

- *Input data:* A natural collection $\{\mathcal{F}(M), Q_M, \omega_M\}_{M\in\mathbf{Loc}_m}$ of free BV theories [Costello/Gwilliam], i.e. $(\mathcal{F}(M), Q_M)$ is a complex of differential operators and ω_M is a (-1)-shifted symplectic structure.

- *Central hypothesis:* Green-hyperbolic complexes, i.e. there exists (pseudo-)natural family of *retarded/advanced Green's homotopies*

$$\Big\{\Lambda_K^\pm \in \big[\mathcal{F}_K(M), \mathcal{F}_{J_M^\pm(K)}(M)\big]^{-1} \,:\, \partial\Lambda_K^\pm = \mathrm{incl}\Big\}_{K\subseteq M \text{ compact}}$$

Thm: [Benini/Musante/AS] One can construct from this data an AQFT $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_m, \perp)^{\mathrm{hoCauchy}}$.

Construction of free (non-interacting) QFTs on $\mathbf{Loc}_m$

- *Input data:* A natural collection $\{\mathcal{F}(M), Q_M, \omega_M\}_{M\in\mathbf{Loc}_m}$ of free BV theories [Costello/Gwilliam], i.e. $(\mathcal{F}(M), Q_M)$ is a complex of differential operators and ω_M is a (-1)-shifted symplectic structure.

- *Central hypothesis:* Green-hyperbolic complexes, i.e. there exists (pseudo-)natural family of *retarded/advanced Green's homotopies*

$$\Big\{\Lambda_K^\pm \in \big[\mathcal{F}_K(M), \mathcal{F}_{J_M^\pm(K)}(M)\big]^{-1} \,:\, \partial\Lambda_K^\pm = \mathrm{incl}\Big\}_{K\subseteq M \text{ compact}}$$

Thm: [Benini/Musante/AS] One can construct from this data an AQFT $\mathfrak{A}\in\mathbf{AQFT}(\mathbf{Loc}_m,\perp)^{\mathrm{hoCauchy}}$.

Ex: Linear Yang-Mills theory [Benini/Bruinsma/AS]

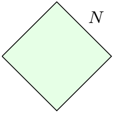
$$\begin{array}{ccccccc}
\overset{(-1)}{\Omega^0_K(M)} & \xrightarrow{\ \mathrm{d}\ } & \overset{(0)}{\Omega^1_K(M)} & \xrightarrow{\ \delta\mathrm{d}\ } & \overset{(1)}{\Omega^1_K(M)} & \xrightarrow{\ \delta\ } & \overset{(2)}{\Omega^0_K(M)} \\
\downarrow\subseteq & \swarrow{\scriptstyle \delta G^\pm_\Box} & \downarrow\subseteq & \swarrow{\scriptstyle G^\pm_\Box} & \downarrow\subseteq & \swarrow{\scriptstyle \mathrm{d}G^\pm_\Box} & \downarrow\subseteq \\
\Omega^0_{J^\pm_M(K)}(M) & \xrightarrow[\ \mathrm{d}\ ]{} & \Omega^1_{J^\pm_M(K)}(M) & \xrightarrow[\ \delta\mathrm{d}\ ]{} & \Omega^1_{J^\pm_M(K)}(M) & \xrightarrow[\ \delta\ ]{} & \Omega^0_{J^\pm_M(K)}(M)
\end{array}$$

Comparison to factorization algebras (à la [Costello/Gwilliam])

◇ Time-orderable prefactorization algebras on $\mathbf{Loc}_m$ [Benini/Perin/AS]:

Comparison to factorization algebras (à la [Costello/Gwilliam])

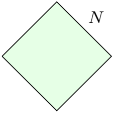
◇ Time-orderable prefactorization algebras on $\mathbf{Loc}_m$ [Benini/Perin/AS]:



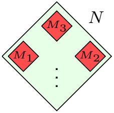
$$\xmapsto{\text{tPFA}} \quad \underbrace{\mathfrak{F}(N) \in \mathbf{T}}_{\text{no algebra structure!!!}}$$

Comparison to factorization algebras (à la [Costello/Gwilliam])

◇ Time-orderable prefactorization algebras on $\mathbf{Loc}_m$ [Benini/Perin/AS]:



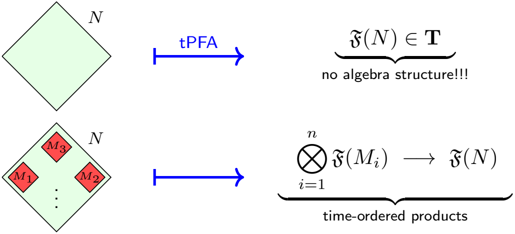
$\overset{\text{tPFA}}{\longmapsto}$ $\underbrace{\mathfrak{F}(N) \in \mathbf{T}}_{\text{no algebra structure!!!}}$



$\longmapsto$ $\underbrace{\bigotimes_{i=1}^{n} \mathfrak{F}(M_i) \longrightarrow \mathfrak{F}(N)}_{\text{time-ordered products}}$

Comparison to factorization algebras (à la [Costello/Gwilliam])

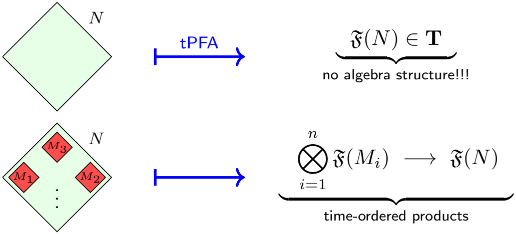
◇ Time-orderable prefactorization algebras on $\mathbf{Loc}_m$ [Benini/Perin/AS]:



◇ With some Lorentzian geometry, one shows that there exists an operad morphism $\Phi : \mathrm{t}\mathcal{P}_{\mathbf{Loc}_m} \to \mathcal{O}_{(\mathbf{Loc}_m,\perp)}$ to the AQFT operad.

Comparison to factorization algebras (à la [Costello/Gwilliam])

◇ Time-orderable prefactorization algebras on $\mathbf{Loc}_m$ [Benini/Perin/AS]:



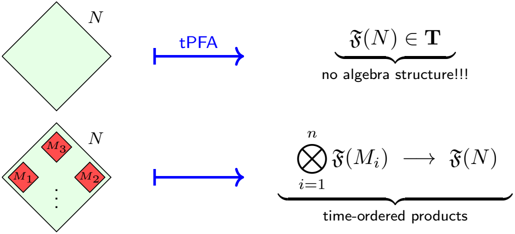
◇ With some Lorentzian geometry, one shows that there exists an operad morphism $\Phi : \mathrm{t}\mathcal{P}_{\mathbf{Loc}_m} \to \mathcal{O}_{(\mathbf{Loc}_m,\perp)}$ to the AQFT operad.

Thm: [Benini/Perin/AS] For target $\mathbf{T}$ = cocomplete SM 1-category, we have an equivalence of categories

$$\Phi_! \;:\; \mathbf{tPFA}_m^{\mathrm{Cauchy},\mathrm{add}} \;\underset{\sim}{\rightleftarrows}\; \mathbf{AQFT}(\mathbf{Loc}_m,\perp)^{\mathrm{Cauchy},\mathrm{add}} \;:\; \Phi^*$$

Comparison to factorization algebras (à la [Costello/Gwilliam])

- Time-orderable prefactorization algebras on $\mathbf{Loc}_m$ [Benini/Perin/AS]:



- With some Lorentzian geometry, one shows that there exists an operad morphism $\Phi : \mathrm{t}\mathcal{P}_{\mathbf{Loc}_m} \to \mathcal{O}_{(\mathbf{Loc}_m,\perp)}$ to the AQFT operad.

Thm: [Benini/Perin/AS] For target $\mathbf{T}$ = cocomplete SM 1-category, we have an equivalence of categories

$$\Phi_! \,:\, \mathbf{tPFA}_m^{\mathrm{Cauchy},\mathrm{add}} \; \underset{\sim}{\rightleftarrows} \; \mathbf{AQFT}(\mathbf{Loc}_m,\perp)^{\mathrm{Cauchy},\mathrm{add}} \,:\, \Phi^*$$

- *Open problem:* Generalization to $\mathbf{T}$ = SM ∞-category, in particular $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$? In this case there are so far only example-based comparisons [Gwilliam/Rejzner, Benini/Musante/AS].

Comparison to functorial field theories (à la [Stolz/Teichner, ...])

- ◇ Working with globally hyperbolic Lorentzian manifolds and Cauchy surfaces, all bordisms are cylinders $M \cong \mathbb{R} \times \Sigma$, but with rich geometry!

Comparison to functorial field theories (à la [Stolz/Teichner, . . .])

◇ Working with globally hyperbolic Lorentzian manifolds and Cauchy surfaces, all bordisms are cylinders $M \cong \mathbb{R} \times \Sigma$, but with rich geometry!

◇ *Conjecture:* Consider the subcategory $\mathbf{Cau}_m \subseteq \mathbf{Loc}_m$ given by all objects, but only Cauchy morphisms. I believe that its localization

$$\mathbf{Cau}_m[\mathrm{Cauchy}^{-1}] \simeq \mathbf{LBord}_m[\mathrm{All}^{-1}]$$

is equivalent to a Stolz-Teichner-style globally hyperbolic Lorentzian bordism category, localized at all bordisms.

Comparison to functorial field theories (à la [Stolz/Teichner, ...])

◇ Working with globally hyperbolic Lorentzian manifolds and Cauchy surfaces, all bordisms are cylinders $M \cong \mathbb{R} \times \Sigma$, but with rich geometry!

◇ *Conjecture:* Consider the subcategory $\mathbf{Cau}_m \subseteq \mathbf{Loc}_m$ given by all objects, but only Cauchy morphisms. I believe that its localization

$$\mathbf{Cau}_m[\mathrm{Cauchy}^{-1}] \simeq \mathbf{LBord}_m[\mathrm{All}^{-1}]$$

is equivalent to a Stolz-Teichner-style globally hyperbolic Lorentzian bordism category, localized at all bordisms.

◇ *Implication:* Each $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_m, \perp)^W$ has an underlying representation of the Lorentzian bordisms that captures time evolution, but ignores spatial locality associated with non-Cauchy morphisms $f : M \to N$.

Comparison to functorial field theories (à la [Stolz/Teichner, ...])

◇ Working with globally hyperbolic Lorentzian manifolds and Cauchy surfaces, all bordisms are cylinders $M \cong \mathbb{R} \times \Sigma$, but with rich geometry!

◇ *Conjecture:* Consider the subcategory $\mathbf{Cau}_m \subseteq \mathbf{Loc}_m$ given by all objects, but only Cauchy morphisms. I believe that its localization

$$\mathbf{Cau}_m[\mathrm{Cauchy}^{-1}] \simeq \mathbf{LBord}_m[\mathrm{All}^{-1}]$$

is equivalent to a Stolz-Teichner-style globally hyperbolic Lorentzian bordism category, localized at all bordisms.

◇ *Implication:* Each $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_m, \perp)^W$ has an underlying representation of the Lorentzian bordisms that captures time evolution, but ignores spatial locality associated with non-Cauchy morphisms $f : M \to N$.

Prop: [Bunk/MacManus/AS; work in progress] The above holds true in spacetime dimension $m = 1$. (...and quite likely also in general dimension)

Comparison to functorial field theories (à la [Stolz/Teichner, ...])

- Working with globally hyperbolic Lorentzian manifolds and Cauchy surfaces, all bordisms are cylinders $M \cong \mathbb{R} \times \Sigma$, but with rich geometry!

- *Conjecture:* Consider the subcategory $\mathbf{Cau}_m \subseteq \mathbf{Loc}_m$ given by all objects, but only Cauchy morphisms. I believe that its localization

$$\mathbf{Cau}_m[\mathrm{Cauchy}^{-1}] \simeq \mathbf{LBord}_m[\mathrm{All}^{-1}]$$

  is equivalent to a Stolz-Teichner-style globally hyperbolic Lorentzian bordism category, localized at all bordisms.

- *Implication:* Each $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_m, \perp)^W$ has an underlying representation of the Lorentzian bordisms that captures time evolution, but ignores spatial locality associated with non-Cauchy morphisms $f : M \to N$.

Prop: [Bunk/MacManus/AS; work in progress] The above holds true in spacetime dimension $m = 1$. (...and quite likely also in general dimension)

- *Open problem:* What corresponds on the FFT side to the additional AQFT structure given by spatial locality? Is this related to extended field theories?

Future direction: Non-affine AQFTs

◇ In examples arising in physics, one typically has that

$$\mathfrak{A}(M) \,=\, \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathsf{uAs}}\big(\mathbf{Ch}_{\mathbb{K}}\big)$$

Future direction: Non-affine AQFTs

- In examples arising in physics, one typically has that

$$\mathfrak{A}(M) \,=\, \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathsf{uAs}}\big(\mathbf{Ch}_{\mathbb{K}}\big)$$

- *Well-known problem:* Interesting derived stacks are almost never affine!
  Example: Classifying stack $\mathrm{B}G = [*/G]$ for G reductive affine group scheme
  $\rightsquigarrow \mathcal{O}(\mathrm{B}G) \simeq N^\bullet(G,\mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ forgets the group

Future direction: Non-affine AQFTs

- In examples arising in physics, one typically has that

$$\mathfrak{A}(M) = \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \in \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{Ch}_{\mathbb{K}})$$

- *Well-known problem:* Interesting derived stacks are almost never affine! Example: Classifying stack $\mathrm{B}G = [*/G]$ for G reductive affine group scheme $\rightsquigarrow \mathcal{O}(\mathrm{B}G) \simeq N^\bullet(G,\mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ forgets the group

- *Way out:* [CPTVV] Assign instead quantizations of dg-categories of modules

$$\boldsymbol{\mathfrak{A}}(M) = \mathrm{QCoh}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \in \mathbf{Alg}_{\mathbb{E}_0}(\mathbf{dgCat}_{\mathbb{K}})$$

Future direction: Non-affine AQFTs

- In examples arising in physics, one typically has that

$$\mathfrak{A}(M) \,=\, \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{Ch}_{\mathbb{K}})$$

- *Well-known problem:* Interesting derived stacks are almost never affine!
  Example: Classifying stack $\mathrm{B}G = [*/G]$ for G reductive affine group scheme
  $\rightsquigarrow \mathcal{O}(\mathrm{B}G) \simeq N^\bullet(G,\mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ forgets the group

- *Way out:* [CPTVV] Assign instead quantizations of dg-categories of modules

$$\mathfrak{A}(M) \,=\, \mathrm{QCoh}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathbb{E}_0}(\mathbf{dgCat}_{\mathbb{K}})$$

Def: A non-affine AQFT is a $\mathbf{dgCat}_{\mathbb{K}}$-valued algebra $\mathfrak{A} \in \mathbf{Alg}_{\mathcal{P}_{(\mathbf{C},\perp)}}(\mathbf{dgCat}_{\mathbb{K}})$ over the factor $\mathcal{P}_{(\mathbf{C},\perp)}$ of the AQFT operad $\mathcal{O}_{(\mathbf{C},\perp)} = \mathcal{P}_{(\mathbf{C},\perp)} \otimes \mathsf{uAs}$.

Future direction: Non-affine AQFTs

- In examples arising in physics, one typically has that

$$\mathfrak{A}(M) \,=\, \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathsf{uAs}}\big(\mathbf{Ch}_{\mathbb{K}}\big)$$

- *Well-known problem:* Interesting derived stacks are almost never affine!
  Example: Classifying stack $\mathrm{B}G = [*/G]$ for G reductive affine group scheme
  $\rightsquigarrow \mathcal{O}(\mathrm{B}G) \simeq N^{\bullet}(G,\mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ forgets the group

- *Way out:* [CPTVV] Assign instead quantizations of dg-categories of modules

$$\mathfrak{A}(M) \,=\, \mathrm{QCoh}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathbb{E}_0}\big(\mathbf{dgCat}_{\mathbb{K}}\big)$$

Def: A non-affine AQFT is a $\mathbf{dgCat}_{\mathbb{K}}$-valued algebra $\mathfrak{A} \in \mathbf{Alg}_{\mathcal{P}_{(\mathbf{C},\perp)}}(\mathbf{dgCat}_{\mathbb{K}})$ over the factor $\mathcal{P}_{(\mathbf{C},\perp)}$ of the AQFT operad $\mathcal{O}_{(\mathbf{C},\perp)} = \mathcal{P}_{(\mathbf{C},\perp)} \otimes \mathsf{uAs}$.

- The formal theory of such non-affine AQFTs was studied in a simpler 2-categorical context (replace $\mathbf{dgCat}_{\mathbb{K}}$ by $\mathbf{Pr}_{\mathbb{K}}$) by [Benini/Perin/AS/Woike].

Future direction: Non-affine AQFTs

◇ In examples arising in physics, one typically has that

$$\mathfrak{A}(M) \,=\, \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathsf{uAs}}(\mathbf{Ch}_{\mathbb{K}})$$

◇ *Well-known problem:* Interesting derived stacks are almost never affine!
Example: Classifying stack $\mathrm{B}G = [*/G]$ for G reductive affine group scheme
$\rightsquigarrow \mathcal{O}(\mathrm{B}G) \simeq N^\bullet(G,\mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ forgets the group

◇ *Way out:* [CPTVV] Assign instead quantizations of dg-categories of modules

$$\mathfrak{A}(M) \,=\, \mathrm{QCoh}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathbb{E}_0}(\mathbf{dgCat}_{\mathbb{K}})$$

Def: A non-affine AQFT is a $\mathbf{dgCat}_{\mathbb{K}}$-valued algebra $\mathfrak{A} \in \mathbf{Alg}_{\mathcal{P}_{(\mathbf{C},\perp)}}(\mathbf{dgCat}_{\mathbb{K}})$ over the factor $\mathcal{P}_{(\mathbf{C},\perp)}$ of the AQFT operad $\mathcal{O}_{(\mathbf{C},\perp)} = \mathcal{P}_{(\mathbf{C},\perp)} \otimes \mathsf{uAs}$.

◇ The formal theory of such non-affine AQFTs was studied in a simpler 2-categorical context (replace $\mathbf{dgCat}_{\mathbb{K}}$ by $\mathbf{Pr}_{\mathbb{K}}$) by [Benini/Perin/AS/Woike].

Ex: (i) Orbifold σ-models with fields $\phi : M \to [X/G_{\mathrm{finite}}]$ [Benini/Perin/AS/Woike]

Future direction: Non-affine AQFTs

◇ In examples arising in physics, one typically has that

$$\mathfrak{A}(M) \,=\, \mathcal{O}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathsf{uAs}}\big(\mathbf{Ch}_{\mathbb{K}}\big)$$

◇ *Well-known problem:* Interesting derived stacks are almost never affine!
Example: Classifying stack $\mathrm{B}G = [*/G]$ for G reductive affine group scheme
$\rightsquigarrow \mathcal{O}(\mathrm{B}G) \simeq N^{\bullet}(G,\mathbb{K}) \simeq \mathbb{K} = \mathcal{O}(*)$ forgets the group

◇ *Way out:* [CPTVV] Assign instead quantizations of dg-categories of modules

$$\mathfrak{A}(M) \,=\, \mathrm{QCoh}\Big(\text{derived moduli stack of fields}\Big)_{\hbar} \,\in\, \mathbf{Alg}_{\mathbb{E}_0}\big(\mathbf{dgCat}_{\mathbb{K}}\big)$$

Def: A non-affine AQFT is a $\mathbf{dgCat}_{\mathbb{K}}$-valued algebra $\mathfrak{A} \in \mathbf{Alg}_{\mathcal{P}_{(\mathbf{C},\perp)}}(\mathbf{dgCat}_{\mathbb{K}})$ over the factor $\mathcal{P}_{(\mathbf{C},\perp)}$ of the AQFT operad $\mathcal{O}_{(\mathbf{C},\perp)} = \mathcal{P}_{(\mathbf{C},\perp)} \otimes \mathsf{uAs}$.

◇ The formal theory of such non-affine AQFTs was studied in a simpler 2-categorical context (replace $\mathbf{dgCat}_{\mathbb{K}}$ by $\mathbf{Pr}_{\mathbb{K}}$) by [Benini/Perin/AS/Woike].

Ex: (i) Orbifold σ-models with fields $\phi : M \to [X/G_{\mathrm{finite}}]$ [Benini/Perin/AS/Woike]

(ii) Non-Abelian Yang-Mills theory on spatial lattices [Benini/Pridham/AS]