# String-net methods for CFT correlators

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Based on work with Jürgen Fuchs and Yang Yang

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# Overview

# Chapter 1

Introduction: two-dimensional conformal field theory

# Two-dimensional conformal field theories

- Infinite-dimensional algebra of local symmetries: vertex algebra $\mathcal{V}$
- Representation category
- Chiral conformal field theory: conformal blocks:
  - Defined on complex curves
  - Solutions of chiral Ward identities
  - Multivalued functions of position of field insertions and complex structure: monodromies
  - Factorization

# Two-dimensional conformal field theories

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  - Defined on complex curves
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  - Multivalued functions of position of field insertions and complex structure: monodromies
  - Factorization
- Full local conformal field theory: consistent systems of correlators
  - Defined for conformal surfaces with boundaries and line defects
  - Specify field content
  - Find correlators: specific conformal blocks that are single valued and obey sewing constraints

# Towards modular functors

Chiral CFT:

Vertex algebra $\mathcal{V}$ $\longrightarrow$ $\mathcal{C} := \mathrm{Rep}(\mathcal{V})$

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Rational CFT:
$\mathcal{C}$ is a modular fusion category:

- $k$-linear semisimple abelian category
- monoidal category
- rigid duals
- braiding, non-degenerate

# Towards modular functors

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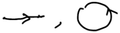
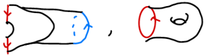
$\mathcal{C}$ is a modular fusion category:

Encode monodromies of conformal blocks:
Open closed modular functor (examples will be constructed in this talk)

$BL_{\mathcal{C}} : \mathrm{Bord}^{\mathrm{or},\sqcup}_{2,o|c} \longrightarrow \mathrm{Prof}_{\mathbb{k}}^{\times}$

- obj: $\emptyset$, , ... — A, B, C, ... $\mathbb{k}$-fin. Cat
- 1-mor: , ... — $A \xrightarrow{P} B : A^{op} \times B \to \mathrm{Vect}_{\mathbb{k}}$
- 2-mor: homeo / homotopy rel. param. — nat

## Towards modular functors

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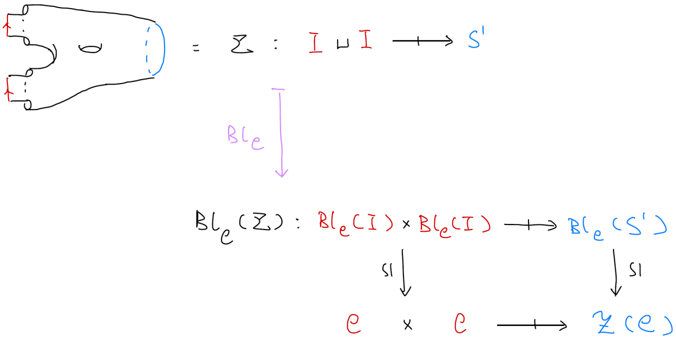
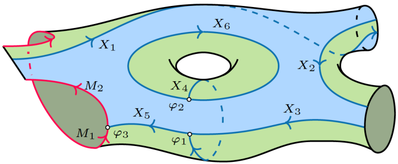
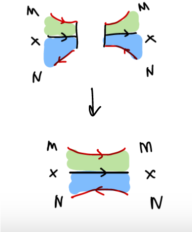


# Full CFT: worldsheet $\mathcal{S}$ with topological defects

Example: world sheet with one sewing interval and two sewing circles:



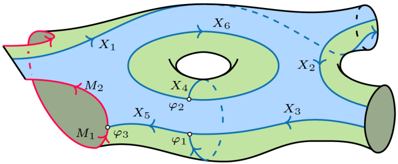




Fix modular tensor category $\mathcal{C}$. Decoration data:

- Special symmetric Frobenius algebras in $\mathcal{C}$ (here two phases, indicated in blue and green)
- Line defects: bimodules
- Point defects: bimodule morphisms

# Full CFT: field objects and correlators

A world sheet $\mathcal{S}$ with one sewing interval and two sewing circles:




Expected field objects

- Interval:

$$\mathrm{Bl}_{\mathcal{C}}(I) = \mathcal{C}$$

Boundary field: $\mathbb{F}(b_1) := \underline{\mathrm{Hom}}(M_2, X_1 \otimes_A M_1)$

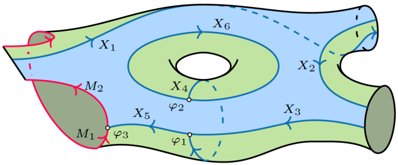
- Circle:

$$\mathrm{Bl}_{\mathcal{C}}(S^1) = \mathcal{Z}(\mathcal{C})$$

Bulk field for lower circle: $\mathbb{F}(b_3) := \underline{\mathrm{Nat}}(G^{X_2}, G^{X_3})$
with $G^X := X \otimes_A -$.

# Full CFT: field objects and correlators

A world sheet $\mathcal{S}$ with one sewing interval and two sewing circles:




Correlator for $\mathcal{S}$:

$$\mathrm{Cor}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) := \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}(b_1), \mathbb{F}(b_2) \times \mathbb{F}(b_3))$$

with $\Sigma_{\mathcal{S}}$ the underlying surface

Conditions on these elements:

- Compatible with sewing
- Invariant under action of mapping class group

$$\mathrm{MCG}(\mathcal{S}) \subset \mathrm{MCG}(\Sigma_{\mathcal{S}})$$

on $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$

# Chapter 2

## Graphical calculus for pivotal bicategories

Input for construction of modular functor:
"String net modular functor is a globalized version of graphical calculus"

# Bicategory

## Definition

A bicategory $\mathcal{B}$ is a category weakly enriched in *Cat*:

- $a, b, c \in \mathrm{Ob}(\mathcal{B})$
- for all pairs $a, b$ a hom-category $\mathcal{B}(a, b) = \mathrm{Hom}_{\mathcal{B}}(a, b)$.
  - Objects: 1-morphisms, morphisms: 2-morphisms of $\mathcal{B}$
  - $k$-linear in this talk.
  - horizontal composition, units

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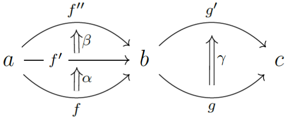
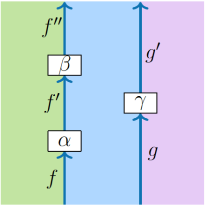
Pasting diagram

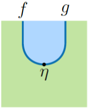



String diagram



# Adjoints in bicategories, pivotal bicategories

Two 1-morphisms $f \in \mathcal{B}(a, b)$ and $g \in \mathcal{B}(b, a)$, together with counit and unit

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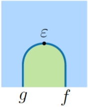


$\eta : 1_a \to g \cdot f$ and $\epsilon : g \cdot f \to 1_b$

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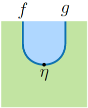


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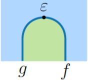
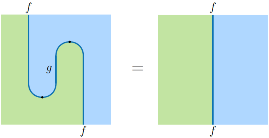



$$\eta : 1_a \to g \cdot f \quad \text{and} \quad \epsilon : g \cdot f \to 1_b$$

satisfying zig-zag relations:

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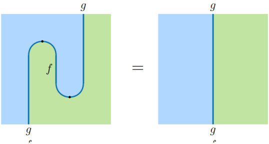


are said to be adjoints:

$$f \dashv g \;\Leftrightarrow\; f = {}^{\vee}g \;\Leftrightarrow\; g = f^{\vee}$$

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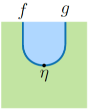


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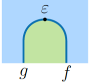
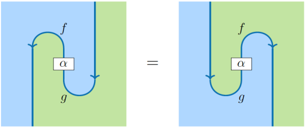



$\eta : 1_a \to g \cdot f$ and $\epsilon : g \cdot f \to 1_b$

## Definition

1. A pivotal structure on a bicategory $\mathcal{B}$ with fixed left and right duals is an identity component pseudonatural transformation $\mathrm{id}_{\mathcal{B}} \to (-)^{\vee\vee}$.
2. A strictly pivotal bicategory is a pivotal bicategory for which the double dual is the identity, $\mathrm{id}_{\mathcal{B}} = (-)^{\vee\vee}$.

Consequence:

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# Examples, graphical calculus

## Examples

A pivotal tensor category $\mathcal{C}$ leads to two pivotal bicategories:

1. Delooping $B\mathcal{C}$: single object $*$ with $\mathrm{End}_{B\mathcal{C}}(*) = \mathcal{C}$.
2. $\mathcal{F}r_{\mathcal{C}}$: objects are simple special symmetric Frobenius algebras $A, B, C, \ldots$. Morphism categories $\mathcal{F}r_{\mathcal{C}}(A, B) = A\text{-mod-}B$ (bimodules)

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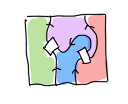
## Remark

Graphical calculus

bicategory  2-framed, progressive

pivotal bicategory  drop progressive, drop 2-framed

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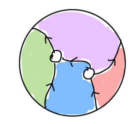


oriented

# Formalization of graphical calculus for a pivotal bicategory

Formulate graphical calculus as a symmetric monoidal functor

$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_{k}^{\otimes}$$

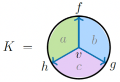
# Formalization of graphical calculus for a pivotal bicategory

Formulate graphical calculus as a symmetric monoidal functor

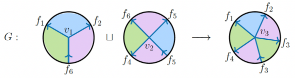
$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_k^{\otimes}$$

Monoidal category $\mathrm{Corollas}_{\mathcal{B}}^{\sqcup}$ of corollas:

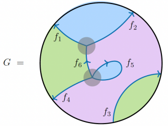
- Objects: $\emptyset$,



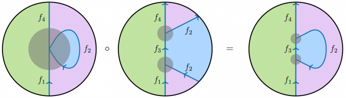

- Morphisms




given by



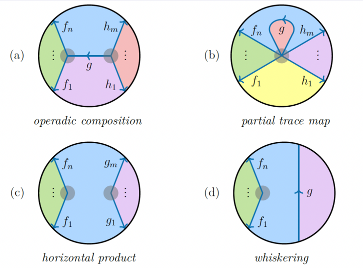

- Composition



# Generation of corollas

## Proposition

*Any non-trivial morphism in* $\mathrm{Corollas}_{\mathcal{B}}^{\sqcup}$ *can be decomposed into a finite disjoint union of partial compositions of morphisms of the following types:*

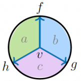



Call $\mathrm{Corollas}_{\mathcal{B}}^{\mathsf{conn}}$ the subcategory of $\mathrm{Corollas}_{\mathcal{B}}$ with the same objects, but morphisms only generated by (a) and (b).

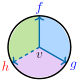
Set up the symmetric monoidal functor expressing graphical calculus:

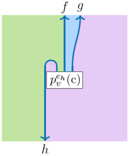
$$\mathrm{GCal}_{\mathcal{B}} : \ \mathrm{Corollas}_{\mathcal{B}}^{\sqcup} \longrightarrow \mathrm{vect}_k^{\otimes}$$

# Graphical calculus on objects

To the corolla $K =$ 

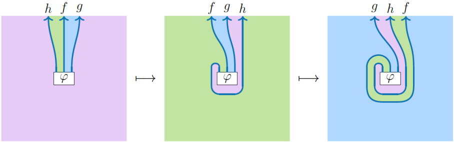
we associate a vector space $H_v$ as follows:

Any polarization 

yields a space of morphisms $=$ 


Pivotal structure allows to relate morphisms for different polarizations by unique isomorphism

Example:

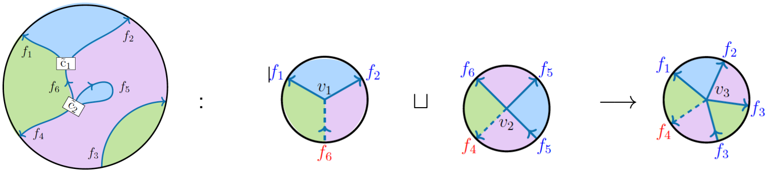



Vector space $H_v$ assigned to corolla

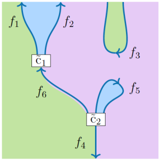
$=$ the limit over this diagram of vector spaces.

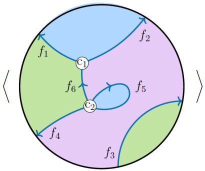
# Graphical calculus on morphisms

To find the linear map for the morphism of corollas




pick polarizations and consider the morphism given by the evaluation of progressive diagram




We write $c_3 =$ 

# Functoriality of the graphical calculus

Lax functor $F : \mathcal{B} \to \mathcal{B}'$ between bicategories $\mathcal{B}, \mathcal{B}'$:

- On objects $F : a \mapsto Fa$
- On Hom-categories $\mathcal{B}(a,b) \mapsto \mathcal{B}'(Fa,Fb)$ with natural transformations

$$\begin{array}{ccc}
\mathcal{B}(a,b) \times \mathcal{B}(b,c) & \xrightarrow{\ \star\ } & \mathcal{B}(a,c) \\
{\scriptstyle F\times F}\downarrow & \overset{F^{(2)}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle F} \\
\mathcal{B}'(Fa,Fb) \times \mathcal{B}'(Fb,Fc) & \xrightarrow[\ \star'\ ]{} & \mathcal{B}'(Fa,Fc)
\end{array}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{\ \mathrm{id}_a\ } & \mathcal{B}(a,a) \\
 & \overset{F^{(0)}}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle F} \\
 & \searrow{\scriptstyle \mathrm{id}_{Fa}} & \mathcal{B}'(Fa,Fa)
\end{array}$$

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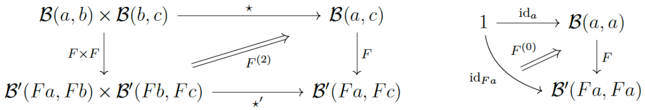



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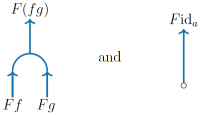
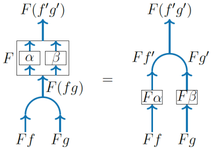
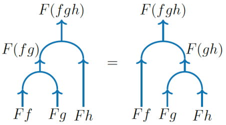
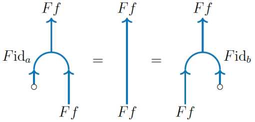



Naturality




Lax Associativity




Lax unity

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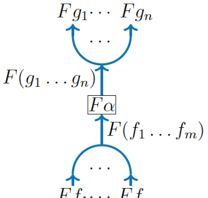
and


Similarly, oplax structure. For a natural transformation

$$\alpha : f_1 \star \ldots \star f_m \Rightarrow g_1 \star \ldots \star g_n$$

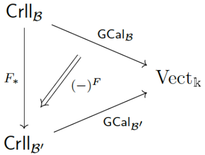
define $F$-conjugate:

$$\alpha^F : \quad Ff_1 \star \cdots \star Ff_m \xrightarrow{F^{(m)}_{f_1,\ldots,f_m}} F(f_1 \star \cdots \star f_m) \xrightarrow{F\alpha} F(g_1 \star \cdots \star g_n) \xrightarrow{F_{(n)\, g_1,\ldots,g_n}} Fg_1 \star \cdots \star Fg_n$$



# When does $F$-conjugation preserve a graphical calculus?

in the sense that $F$-conjugation induces a monoidal natural transformation



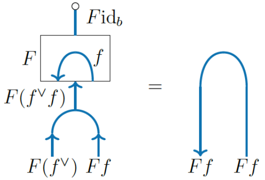

which is given on a corolla $K$ by

$$(-)^F_K : \quad \mathsf{GCal}^{\mathrm{conn}}_{\mathcal{B}}(K) \xrightarrow{p^k_v} \widehat{h^{\mathcal{B}}_v}(k) \xrightarrow{(-)^F} \widehat{h^{\mathcal{B}'}_{v'}}(k) \xrightarrow{(p^k_{v'})^{-1}} \mathsf{GCal}^{\mathrm{conn}}_{\mathcal{B}}(F_*K)$$

# When does $F$-conjugation preserve a graphical calculus?

Optimal case:

- $F$ is rigid, i.e. $F(f^\vee) = F(f)^\vee$ and $F$ preserves the counits of duals:




- And $F$ is a pseudofunctor: $F^{(2)} = F_{(2)}^{-1}$ and $F^{(0)} = F_{(0)}^{-1}$.

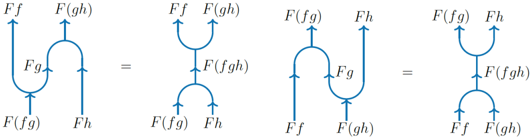
**Theorem**

*A rigid pseudofunctor preserves the graphical calculus.*

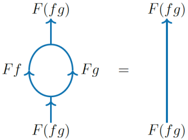
# When does $F$-conjugation preserve a graphical calculus?

## Subptimal case:

- $F$ is rigid
- $F$ is Frobenius

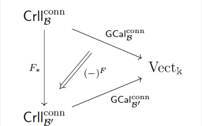



- $F$ is separable



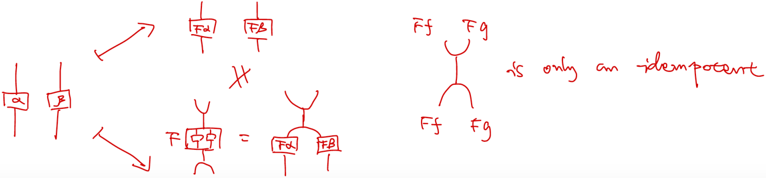

### Theorem

*A rigid Frobenius functor preserves the connected graphical calculus:*




*operadic composition and partial trace preserved, but not whiskering and horizontal product.*

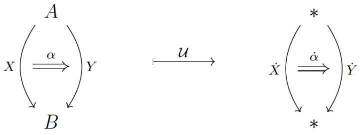
# When does $F$-conjugation preserve a graphical calculus?



# Example of a rigid separable Frobenius functor

$\mathcal{C}$ a pivotal tensor category: using separable special symmetric Frobenius algebras:

$$\mathcal{U}: \ \mathcal{F}r_{\mathcal{C}} \to B\mathcal{C}$$



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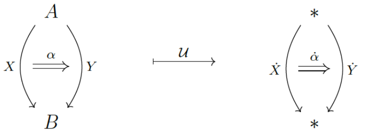
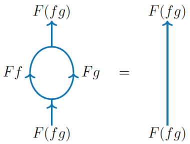
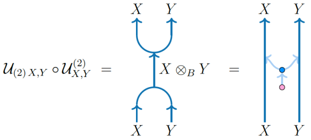



Lax monoidal and opmonoidal constraint

$$\mathcal{U}^{(2)}_{X,Y} : \quad X \otimes Y \longrightarrow X \otimes_B Y \qquad \text{and} \qquad \mathcal{U}_{(2)\,X,Y} : \quad X \otimes_B Y \longrightarrow X \otimes Y$$

defined by splitting of the idempotent



## Chapter 3

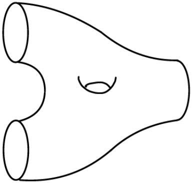
### String-net models for pivotal bicategories

Will be applied to two different pivotal bicategories:

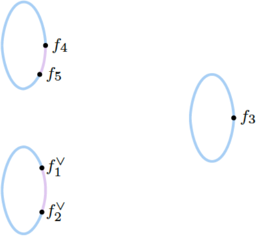
- $\mathcal{B} = B\mathcal{C}$, provides modular functor for conformal blocks
- $\mathcal{B} = \mathcal{F}r_{\mathcal{C}}$ describes world sheets with defects

# String-net models for pivotal bicategories

$\Sigma$ oriented surface



boundary value $b$, e.g.



# String-net models for pivotal bicategories

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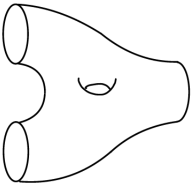

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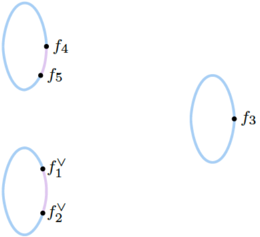
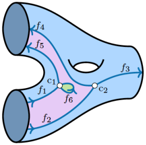
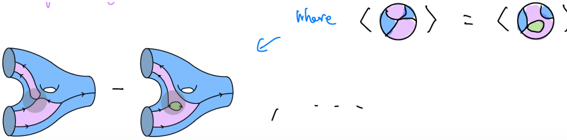



$$\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathrm{b}) := \Bbbk\mathrm{G}(\Sigma, \mathrm{b})/\mathrm{N}(\Sigma, \mathrm{b}) \,,$$

with $\mathbb{K}\mathrm{G}(\Sigma, b)$ the vector space freely generated by labelled surfaces with boundary condition $b$, e.g.




and $N(\Sigma, b)$ the subspace generated by local relations given by graphical calculus:



# String-net spaces as colimits

Set up evaluation functor

$$\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}} : \quad \mathcal{G}\text{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b}) \longrightarrow \mathrm{Vect}_{\Bbbk}$$

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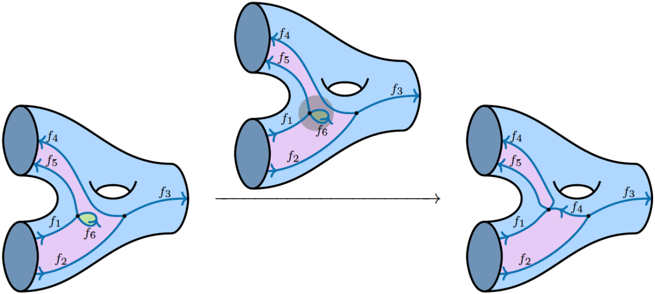
- Objects of $\mathcal{G}\mathrm{raphs}(\Sigma, b)$: partially colored surfaces:
  - patches $\to$ objects of $\mathcal{B}$
  - edges $\to$ 1-morphisms of $\mathcal{B}$
  - vertices are not colored
- Morphisms of $\mathcal{G}\mathrm{raphs}(\Sigma, b)$ are freely generated by e.g.



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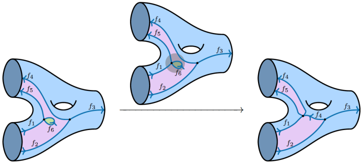



$$\mapsto \bigotimes_{v \in V(\Gamma_1)} H_v^{\mathcal{B}} \longrightarrow \bigotimes_{v \in V(\Gamma_2)} H_v^{\mathcal{B}}$$

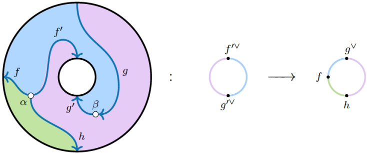
## Theorem

1. *The string net space is a colimit:* $\mathrm{SN}_{\mathcal{B}}^{\circ}(\Sigma, b) = \mathrm{colim}\mathcal{E}_{\mathcal{B}}^{\Sigma,b}$
2. *The mapping class group* $\mathrm{MCG}(\Sigma)$ *acts on the string net space.*
3. *Sewing holds: we have a modular functor.*

Remark:
Generalization to modular functors with values in $\mathrm{Ch}_{\mathbb{K}}$ via homotopy limits.

# Cylinder categories for pointed bicategories

For any closed oriented 1-manifold $\ell$, define a category $\mathrm{Cyl}^{\circ}(\mathcal{B}, \ell)$.



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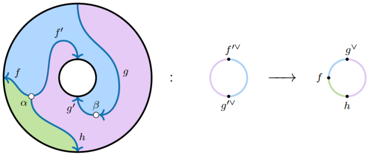


The bicategories $B\mathcal{C}$ and $\mathcal{F}r_{\mathcal{C}}$ are pointed:
Distinguished object $*_{\mathcal{B}} \in \mathcal{B}$: for $\mathcal{F}r_{\mathcal{C}}$ this is the Frobenius algebra $1 \in \mathcal{C}$.

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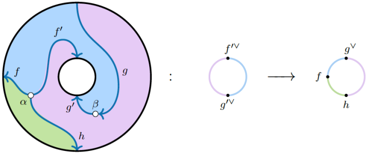
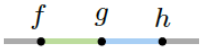



The bicategories $BC$ and $\mathcal{F}r_{\mathcal{C}}$ are pointed:
Distinguished object $*_{\mathcal{B}} \in \mathcal{B}$: for $\mathcal{F}r_{\mathcal{C}}$ this is the Frobenius algebra $1 \in \mathcal{C}$.
E.g. $\ell = I$: label 1-cells adjacent to boundary point by $*_{\mathcal{B}}$:

b $=$ 

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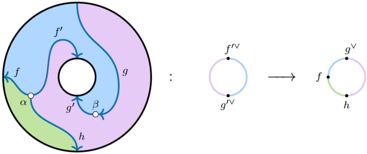


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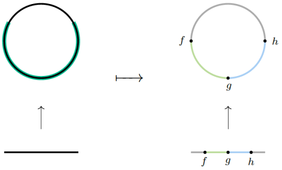


**Remark**

1. Functoriality under embedding of 1-manifolds: symmetric monoidal functor

$$\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, -) : \quad \mathrm{Emb}_1^{\mathrm{or}} \longrightarrow \mathrm{Cat}_{\Bbbk} .$$

2. Induces on $\mathrm{Cyl}^{\circ}(\mathcal{B}, a, I)$ the monoidal structure of $\mathcal{B}(a, a)$.

# Cylinder categories for pivotal fusion categories

$\mathcal{C}$ a pivotal fusion category. Karoubify:

$$\mathrm{Cyl}(\mathcal{C}, I) \cong \mathcal{C} \quad \text{and} \quad \mathrm{Cyl}(\mathcal{C}, S^1) \cong \mathcal{Z}(\mathcal{C})$$

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Geometric embedding induces left adjoint $L$ of $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$:

$$\begin{array}{ccc} \mathrm{Cyl}(\mathcal{C}, I) & \xrightarrow{I \hookrightarrow S^1} & \mathrm{Cyl}(\mathcal{C}, S^1) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{C} & \xrightarrow{L} & \mathcal{Z}(\mathcal{C}) \end{array}$$

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String net space for surface $\Sigma$ with karoubified boundary data:
for idempotent $B : b \to b$ in $\mathrm{Cyl}(\mathcal{B}, *_\mathcal{B}, \Sigma)$, consider subspace

$$\mathrm{SN}_\mathcal{B}(\Sigma, B) \;\subset\; \mathrm{SN}^0(\Sigma, b)$$

invariant under glueing with $B$

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invariant under glueing with $B$
This can be extended to a 3-2-1 TFT equivalent to the Turaev-Viro construction.

# Modular functor

One can prove factorization:

**Theorem 3.23.** *Let* $\Sigma\colon \alpha \sqcup \beta \nrightarrow \beta \sqcup \gamma$ *be a bordism, with* $\alpha, \beta, \gamma \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$. *Then the family*

$$\left\{ s^{\Sigma}_{-,\mathsf{b}_0,\sim} : \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \mathsf{b}_0, \mathsf{b}_0, \sim) \Longrightarrow \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim) \right\}_{\mathsf{b}_0 \in \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \beta)} \tag{3.45}$$

*of natural transformations whose members are given by the sewing of string nets, is dinatural and exhibits the functor* $\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)$ *as the coend*

$$\int^{\mathsf{b} \in \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \beta)} \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \mathsf{b}, \mathsf{b}, \sim) : \quad \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \nrightarrow \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \gamma)\,. \tag{3.46}$$

# Modular functor

and obtains a modular functor with values in profunctors:

**Theorem 3.27.** *Let $(\mathcal{B}, *_{\mathcal{B}})$ be a pointed strictly pivotal bicategory. Then the assignments*

$$\alpha \mapsto \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \qquad \text{and} \qquad \Sigma \mapsto \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \sim) \tag{3.67}$$

*extend to an* open-closed modular functor, *i.e. a symmetric monoidal pseudofunctor*

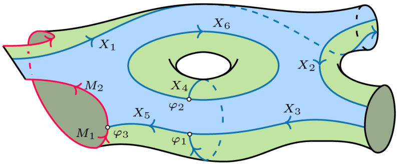
$$\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}} : \quad \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \longrightarrow \mathcal{P}\mathrm{rof}_{\Bbbk} \tag{3.68}$$

*from the symmetric monoidal bicategory of open-closed bordisms to the symmetric monoidal bicategory of* $\Bbbk$*-linear profunctors. Similarly, the Karoubified cylinder categories and string-net spaces give rise to another open-closed modular functor*

$$\mathcal{S}\mathrm{N}_{\mathcal{B}} : \quad \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \longrightarrow \mathcal{P}\mathrm{rof}_{\Bbbk}\,. \tag{3.69}$$

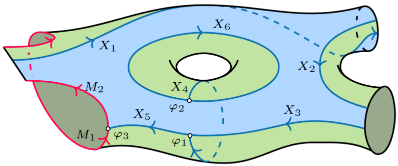
# Chapter 4

Applications to RCFTs with defects: universal correlators




world sheet $\mathcal{S}$ with defects, decoration data in $\mathcal{F}r_{\mathcal{C}}$

# Correlators, finally

$\mathcal{S} =$ 


worldsheet

$\downarrow$

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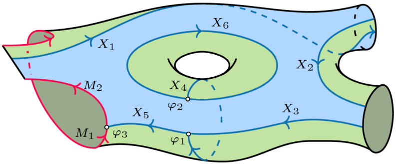
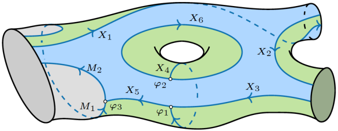



worldsheet

$\downarrow$

$\tilde{\mathcal{S}} =$




complemented worldsheet

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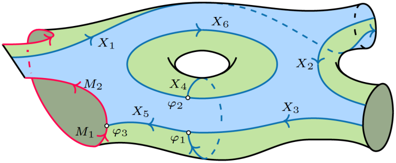


worldsheet

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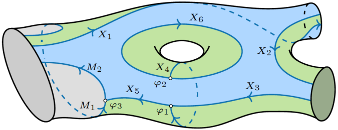


complemented worldsheet

$\downarrow$

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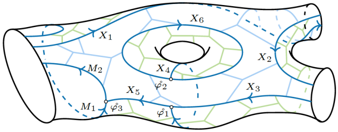


$\in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}b)$

Vertices $\mathcal{U}$-conjugated, Frobenius triangulation

# Field content

General scheme:

$$\mathrm{Cyl}^{\circ}(\mathcal{F}r_{\mathcal{C}},\cdot) \xrightarrow{\mathbb{F}} \mathrm{Cyl}(\mathcal{C},\cdot) \xrightarrow{\cong} \mathcal{C} \text{ or } \mathcal{Z}(\mathcal{C})$$

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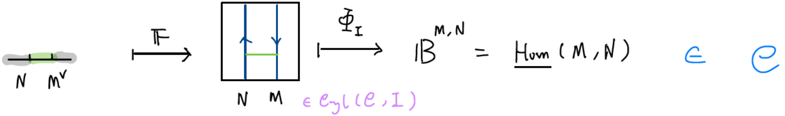
(Generalized) boundary fields:



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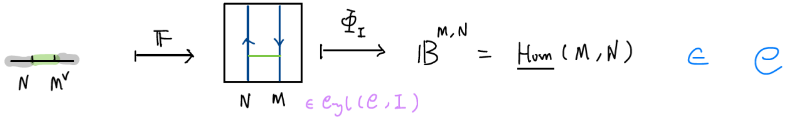
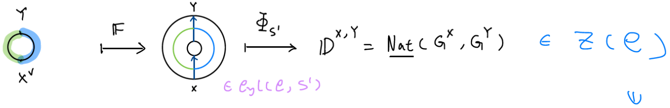
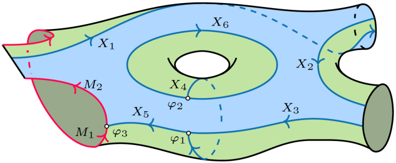



For $A = B$ and $X = Y = A$, get $\mathbb{D}^{A,A} = \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}-A}, \mathrm{id}_{\mathrm{mod}-A}) = Z(A)$

# Quantum world sheet

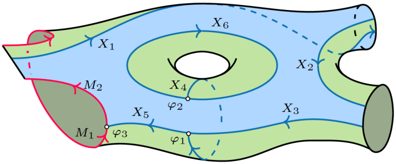
$\mathcal{S} =$ 


worldsheet

$\downarrow$

# Quantum world sheet

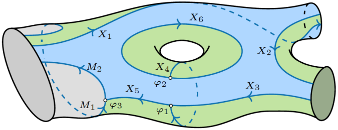
$\mathcal{S}$ =




worldsheet

$\downarrow$

$[\tilde{\mathcal{S}}]$ =



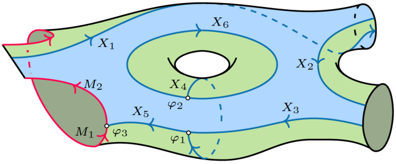

$\in \ \mathrm{SN}^0_{\mathcal{F}r_\mathcal{C}}(\Sigma_\mathcal{S}, \mathbb{F}, b_\mathcal{S})$

quantum world sheet

$\downarrow$

# Quantum world sheet

$\mathcal{S} =$




worldsheet

$\downarrow$

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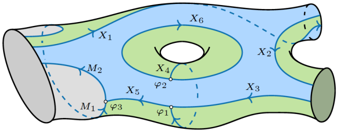
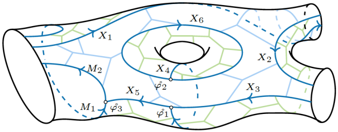



$\in \ \mathrm{SN}^0_{\mathcal{F}r_{\mathcal{C}}}(\Sigma_{\mathcal{S}}, \mathbb{F}, b_{\mathcal{S}})$

quantum world sheet

$\downarrow$

$\mathrm{Cor}_{\mathcal{S}} :=$




$\in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F} b_{\mathcal{S}})$

# Universal correlators

Correlators factor through vector space of quantum world sheets:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\ \delta_{\mathcal{S}}\ } & \mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma) \\
\Big\downarrow \mathrm{Cor}_{\mathcal{S}} & \swarrow \mathrm{UCor}_{\Sigma} & \\
\mathrm{SN}_{\mathcal{C}}(\Sigma) & &
\end{array}
$$

Correlators can be obtained as pullback of universal correlator:

$$(\delta_{\mathcal{S}})^{*}\mathrm{UCor}_{\Sigma} = \mathrm{Cor}_{\mathcal{S}}$$

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$$(\delta_{\mathcal{S}})^{*}\mathrm{UCor}_{\Sigma} = \mathrm{Cor}_{\mathcal{S}}$$

## Remarks

1. Uses that $\mathcal{U} : \mathcal{F}r_{\mathcal{C}} \to B\mathcal{C}$ is a rigid separable Frobenius functor
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# Universal correlators

Correlators factor through vector space of quantum world sheets:

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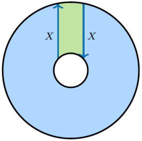
The conformal field theory cannot detect the complete world sheet geometry, only the image in the space of quantum world sheets. The latter is determined by defect data.

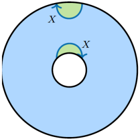
# Mapping class groups of quantum world sheets

$X$ is an invertible topological defect. Then the two worldsheets

$\mathcal{S}_1 :=$




and

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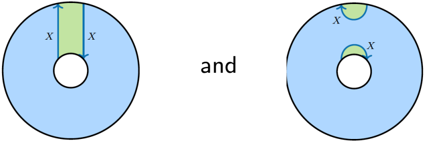
$\mathcal{S}_1 :=$  $=: \mathcal{S}_2$.



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$\mathrm{Map}(\Sigma)$ acts on $\mathrm{SN}_{\mathcal{F}rob(\mathcal{C})}(\Sigma)$ and the appropriate mapping class group for the worldsheet $\mathcal{S}$ with quantum world sheet $[\tilde{\mathcal{S}}] \in \mathrm{SN}_{\mathcal{F}r_\mathcal{C}}(\Sigma)$ is the stabilizer:

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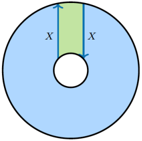
$\mathcal{S}_1 :=$ 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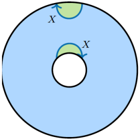

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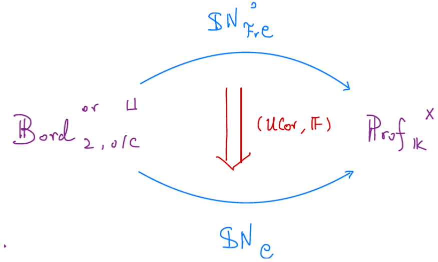
$\widehat{\mathrm{Map}}(\mathcal{S}_1)$ contains Dehn twists while $\mathrm{Map}(\mathcal{S})$ does not.

$\mathrm{Cor}_{\mathcal{S}_1} = \mathrm{Cor}_{\mathcal{S}_2}$ is invariant under $\widehat{\mathrm{Map}}(\mathcal{S}_1)$!

# Schematic overview



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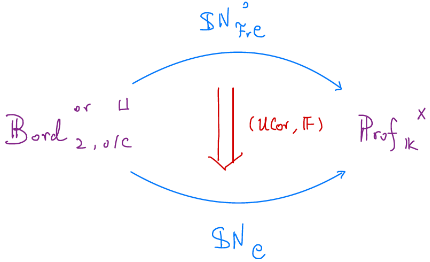

**Remark**

Upgrade to

- Symmetric monoidal double categories
- Symmetric monoidal double functors
- Monoidal vertical transformations

# Summary and outlook

## Summary

- String net constructions are a natural conceptual home for the construction of correlators
- Bicategorical string net construction capture symmetries of CFTs and the observable aspect of the world sheet.

## Outlook

- Beyond semisimplicity
- Beyond rigidity (percolation)
- Construction is set up to go to homotopical versions (3dSYM/VOA correspondence)
- Higher dimensions?