An introduction to decomposition

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An overview of hep-th/0502027, 0502044, 0502053, 0606034, ... (many ...),
& recently arXiv: 2101.11619, 2106.00693, 2107.12386, 2107.13552, 2108.13423,
2204.09117, 2204.13708, 2206.14824, 2208.05982, 2211.14332, & to appear
w/ L. Lin, T. Pantev, D. Robbins, T. Vandermeulen, & many others

My talk today concerns **decomposition**,
a new notion in quantum field theory (QFT).

Briefly, decomposition is the observation that some local QFTs
are secretly equivalent to
sums of other local QFTs, known as ‘universes.’

When this happens, we say the QFT `decomposes.’
Decomposition of the QFT can be applied to give insight
into its properties.

What does it mean for one local QFT to be a sum of other local QFTs?

(Hellerman et al '06)

1) Existence of projection operators

The theory contains topological operators Π_i such that

$$\Pi_i \Pi_j = \delta_{i,j} \Pi_j \qquad \sum_i \Pi_i = 1 \qquad [\Pi_i, \mathcal{O}] = 0$$

Operators Π_i simultaneously diagonalizable; state space = $\mathscr{H} = \oplus_i \mathscr{H}_i$

Correlation functions:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \Pi_i \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_i \langle \left(\Pi_i \mathcal{O}_1\right) \cdots \left(\Pi_i \mathcal{O}_m\right) \rangle = \sum_i \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_m \rangle_i$$

2) Partition functions decompose

$$Z = \sum_{\text{states}} \exp(-\beta H) = \sum_i \sum \exp(-\beta H_i) = \sum_i Z_i$$

(on a connected spacetime)

When does this happen?

There are many examples of decomposition !

Finite gauge theories in 2d (orbifolds): we'll see examples later.

(T Pantev, ES '05; D Robbins, ES, T Vandermeulen '21)

Common thread: a subgroup of the gauge group acts trivially.

Example: If $K \subset \text{center}(\Gamma) \subset \Gamma$ acts trivially, then $[X/\Gamma] = \coprod_{\text{irreps } K} [X/(\Gamma/K)]_{\hat{\omega}}$

Gauge theories:

- 2d $U(1)$ gauge theory with nonmin' charges = sum of $U(1)$ theories w/ min charges (Hellerman et al '06)
- 2d G gauge theory w/ center-invt matter = sum of $G/Z(G)$ theories w/ discrete theta (ES '14)

Ex: $SU(2)$ theory (w/ center-invt matter) = $SO(3)_+ \coprod SO(3)_-$ (w/ same matter)

- 2d pure G Yang-Mills = sum of trivial QFTs indexed by irreps of G (Nguyen, Tanizaki, Unsal '21) (U(1): Cherman, Jacobson '20)

Ex: pure $SU(2) = \coprod_{\text{irreps } SU(2)}$ (sigma model on pt)

- 4d Yang-Mills w/ restriction to instantons of deg' divisible by k (Tanizaki, Unsal '19)

= union of ordinary 4d Yang-Mills w/ different θ angles

More examples ....

There are many examples of decomposition !

More examples :

TFTs: 2d unitary TFTs w/ semisimple local operator algebras decompose to invertibles

(Implicit in Durhuus, Jonsson '93; Moore, Segal '06)

(Also: Komargodski et al '20, Huang et al 2110.02958)

Examples:

- 2d abelian BF theory at level k = disjoint union of k invertibles (sigma models on pts)

(Hellerman, ES, 1012.5999)

- 2d G/G model at level k = disjoint union of invertible theories
  as many as integrable reps of the Kac-Moody algebra

(Komargodski et al 2008.07567)

- 2d Dijkgraaf-Witten = sum of invertible theories, as many as irreps
  (In fact, is a special case of orbifolds discussed later in this talk.)

Sigma models on gerbes = disjoint union of sigma models on spaces w/ B fields

(T Pantev, ES '05)

Solves tech issue w/ cluster decomposition.

What do these examples have in common?....

What do the examples have in common?
When is one local QFT a sum of other local QFTs ?

Answer: in d spacetime dimensions,
a theory decomposes when it has a $(d-1)$-form symmetry.

(2d: Hellerman et al '06;
d>2: Tanizaki-Unsal '19, Cherman-Jacobson '20)

Decomposition & higher-form symmetries go hand-in-hand.

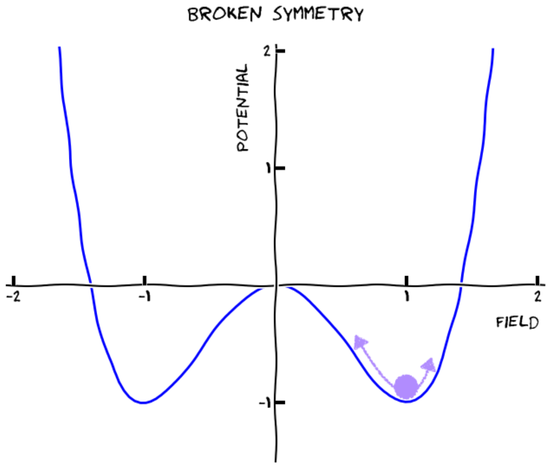
Decomposition ≠ spontaneous symmetry breaking

SSB:

Superselection sectors:

- separated by dynamical domain walls
- only genuinely disjoint in IR
- only one overall QFT

Prototype:

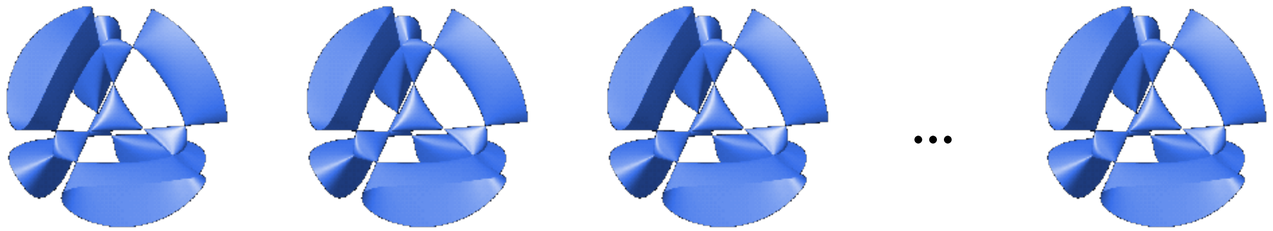


Decomposition:

Universes:

- separated by *non*dynamical domain walls
- disjoint at *all* energy scales
- *multiple* different QFTs present

Prototype:



(see e.g. Tanizaki-Unsal 1912.01033)

Since 2005, decomposition has been checked in many examples in many ways. Examples:

- GLSM's: mirrors, quantum cohomology rings (Coulomb branch) (T Pantev, ES '05; Gu et al '18-'20)
- Orbifolds: partition f'ns, massless spectra, elliptic genera (T Pantev, ES '05; Robbins et al '21)
- Open strings, K theory (Hellerman et al hep-th/0606034)
- Susy gauge theories w/ localization (ES 1404.3986)
- Nonsusy pure Yang-Mills ala Migdal (ES '14; Nguyen, Tanizaki, Unsal '21)
- Adjoint QCD_2 (Komargodski et al '20)
- Numerical checks (lattice gauge thy) (Honda et al '21)
- Versions in d-dim'l theories w/ (d-1)-form symmetries (Tanizaki, Unsal, '19; Cherman, Jacobson '20)

This list is incomplete; apologies to those not listed.

Applications include:

- Sigma models with target stacks & gerbes (T Pantev, ES '05)
- Predictions for Gromov-Witten theory (checked by H-H Tseng, Y Jiang, E Andreini, etc starting '08)
- Nonperturbative constructions of geometries in GLSMs (Caldararu et al 0709.3855, Hori '11, ..., Romo et al '21)
- Elliptic genera (Eager et al '20)
- Anomalies in orbifolds (Robbins et al '21)

In d spacetime dimensions,

a theory decomposes when it has a global $(d-1)$-form symmetry.

Today we'll produce exs by gauging a trivially-acting $(d-2)$-form symmetry

(<-> non-complete charge spectrum)

This is equivalent to

- Theory w/ restriction on instantons
- Sigma models on gerbes

  = fiber bundles with fibers = `groups' of form symmetries $G^{(d-1)} = B^{d-1}G$
- Algebra of topological local operators

Decomposition (into 'universes') relates these pictures.

Examples:

restriction on instantons = "multiverse interference effect"

form symmetry of QFT = translation symmetry along fibers of gerbe

trivial group action b/c $BG = [\text{point}/G]$

Goal for today: a (hopefully pedagogical) introduction to decomposition

Outline:

- **Decomposition in 2d orbifolds, from a perspective that will motivate later cases**

  Global 1-form symmetry from gauging trivially-acting 0-form symmetry

  Aside on gauge theory examples

- Decomposition in 3d orbifolds

  Global 2-form symmetry from gauging trivially-acting 1-form symmetry

- Decomposition in 3d Chern-Simons

  Global 2-form symmetry from gauging trivially-acting 1-form symmetry

Let's first construct a family of examples in $d = 2$ spacetime dimensions.

We'll gauge a noneffectively-acting $(d-2) = 0$-form symmetry,
to get a global 1-form symmetry (& hence a decomposition).

Specifically, consider the orbifold $[X/\Gamma]$, where

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \sim \omega \in H^2(G,K)$$

is a central extension, and K, Γ, G are finite, K abelian, and K acts trivially.
(Decomposition exists more generally, but today I'll stick w/ easy cases.)

The orbifold $[X/\Gamma]$ has a global $BK = K^{(1)}$ symmetry, & should decompose.

I'm going to outline one way to see that

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$$

where $H^2(G,K) \longrightarrow H^2(G,U(1))$, $\omega \mapsto \rho(\omega)$ gives the discrete torsion on universe ρ

Claim: $\mathrm{QFT}([X/\Gamma]) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2,

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) \quad \text{where } Z_{g,h} = \left(g \;\; \blacksquare \;\; \longrightarrow X \right)$$

h

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We need to count commuting pairs of elements in Γ ....

Claim: $\mathrm{QFT}([X/\Gamma]) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

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We need to count commuting pairs of elements in Γ ....

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad \sim\ \omega \in H^2(G,K)$$

Write $\gamma \in \Gamma$ as $\gamma = (g \in G, k \in K)$ where $\gamma_1\gamma_2 = (g_1g_2, k_1k_2\omega(g_1,g_2))$

Then, $\gamma_1\gamma_2 = \gamma_2\gamma_1 \Leftrightarrow g_1g_2 = g_2g_2$ and $\omega(g_1,g_2) = \omega(g_2,g_1)$

commuting pairs in G such that $\omega(g_1,g_2) = \omega(g_2,g_1)$

Restriction on nonperturbative sectors

(In an orbifold, nonperturbative sectors = twisted sectors)

Claim: $\mathrm{QFT}([X/\Gamma]) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$

Let's establish this in partition functions on T^2.

Universally, for any Γ orbifold on T^2, $Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|}\sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X)$

We need to count commuting pairs of elements in Γ .... $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$

These are commuting pairs in G such that $\omega(g_1, g_2) = \omega(g_2, g_1)$

So: $$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|}\sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) = \frac{|K|^2}{|\Gamma|}\sum_{g_1g_2=g_2g_1} \delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)} - 1\right) Z_{g_1,g_2}$$

where we have used $Z_{\gamma_1,\gamma_2} = Z_{g_1,g_2}$ since K acts trivially.

Claim: $$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$$

Let's establish this in partition functions on T^2.

So far: $$Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|}\sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} Z_{\gamma_1,\gamma_2}(X) = \frac{|K|^2}{|\Gamma|}\sum_{g_1g_2=g_2g_1}\delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)}-1\right)Z_{g_1,g_2}$$

Next, write

$$\delta\left(\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)}-1\right) = \frac{1}{|\hat{K}|}\sum_{\rho\in\hat{K}}\frac{\rho\circ\omega(g_1,g_2)}{\rho\circ\omega(g_2,g_1)}$$

where $\rho\circ\omega\in H^2(G,U(1))$ (discrete torsion!)

so that, after rearrangement,

$$Z_{T^2}\left([X/\Gamma]\right) = \frac{|G||K|^2}{|\Gamma||\hat{K}|}\sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho\circ\omega}\right) = \sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho\circ\omega}\right)$$

consistent with decomposition !

Adding the universes projects out some sectors — interference effect.

So far we have demonstrated that for T^2 partition functions,

$$\mathrm{QFT}\left([X/\Gamma]\right) \;=\; \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$$

which is the statement of decomposition in this case ($K \subset \Gamma$ central).

Similar computations can be done at any genus,
and for local operators, etc.

Next, we'll walk through details in a simple example....

To make this more concrete, let's walk through an example,
where everything can be made completely explicit.

Example: Orbifold $[X/D_4]$ in which the $\mathbb{Z}_2$ center acts trivially.

— has $B\mathbb{Z}_2$ (1-form) symmetry

(T Pantev, ES '05)

$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ so this is closely related to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Decomposition predicts

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right)$$

which here means

$$\mathrm{QFT}\left([X/D_4]\right) = \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \mathrm{QFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Let's check this explicitly....

Example, cont'd

$$\text{QFT}\left([X/D_4]\right) \;=\; \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

At the level of operators, one reason for this is that the theory admits projection operators:

Let $\hat{z}$ denote the (dim 0) twist field associated to the trivially-acting $\mathbb{Z}_2$:

$\hat{z}$ obeys $\hat{z}^2 \;=\; 1.$

Using that relation, we form projection operators:

$$\Pi_{\pm} \;=\; \frac{1}{2}(1 \pm \hat{z}) \qquad (= \text{specialization of general formula})$$

$$\Pi_{\pm}^2 = \Pi_{\pm} \qquad \Pi_{\pm}\Pi_{\mp} = 0 \qquad \Pi_+ + \Pi_- = 1$$

Note: untwisted sector lies in both universes; universes = lin' comb's of twisted & untwisted.

Next: compare partition functions....

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

Take the (1+1)-dim'l spacetime to be T^2.

The partition function of any orbifold $[X/\Gamma]$ on T^2 is

$$Z_{T^2}([X/\Gamma]) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \;\; \underset{h}{\square} \;\; \longrightarrow X \right)$$

("twisted sectors")

(Think of $Z_{g,h}$ as sigma model to X with branch cuts g, h.)

We're going to see that

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) + Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

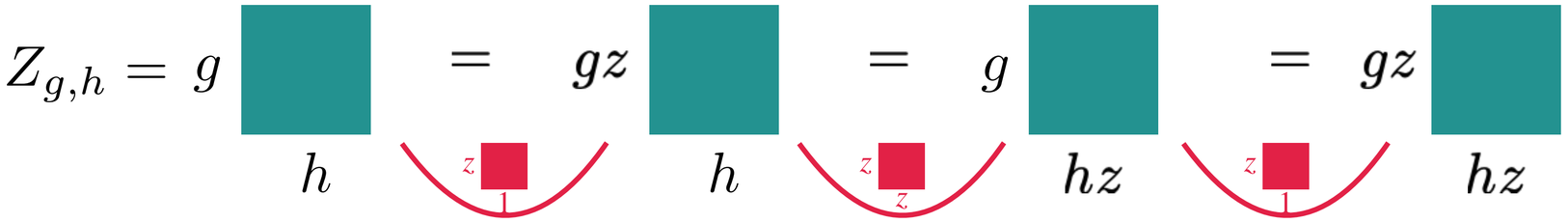
where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\} \qquad \text{where } \overline{a} = \{a, az\} \text{ etc}$$

$$Z_{T^2}\left([X/D_4]\right) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h} \qquad \text{where } Z_{g,h} = \left(g \xrightarrow[h]{} X \right)$$

Since z acts trivially,

$Z_{g,h}$ is symmetric under multiplication by z



This is the $B\mathbb{Z}_2$ 1-form symmetry.

Example, cont'd

Compute the partition function of $[X/D_4]$ (T Pantev, ES '05)

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where z generates the $\mathbb{Z}_2$ center.

$$D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$ where $\overline{a} = \{a, az\}$ etc

$$Z_{T^2}([X/D_4]) = \frac{1}{|D_4|} \sum_{g,h \in D_4,\ gh=hg} Z_{g,h}$$ where $Z_{g,h} = \left(g \, \square_h \longrightarrow X \right)$

Each D_4 twisted sector ($Z_{g,h}$) that appears is the same as a $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ twisted sector,

appearing with multiplicity $|\mathbb{Z}_2|^2 = 4$,

except for the sectors $\overline{a} \, \square_{\overline{b}}$ $\quad \overline{a} \, \square_{\overline{ab}}$ $\quad \overline{b} \, \square_{\overline{ab}}$ which do **not** appear.

Restriction on nonperturbative sectors

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Different theory than $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold

Physics knows when we gauge even a trivially-acting group!

Example, cont'd

Compute the partition function of $[X/D_4]$

(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2 \times \mathbb{Z}_2|}{|D_4|} |\mathbb{Z}_2|^2 \left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Fact: given any one partition function $Z_{T^2}([X/G]) = \frac{1}{|G|} \sum_{gh=hg} Z_{g,h}$

we can multiply in $SL(2,\mathbb{Z})$-invariant phases $\epsilon(g,h)$

to get another consistent partition function (for a different theory)

$$Z' = \frac{1}{|G|} \sum_{gh=hg} \epsilon(g,h)\, Z_{g,h}$$

There is a universal choice of such phases, determined by elements of $H^2(G, U(1))$

This is called "discrete torsion."

Example, cont'd

Compute the partition function of $[X/D_4]$

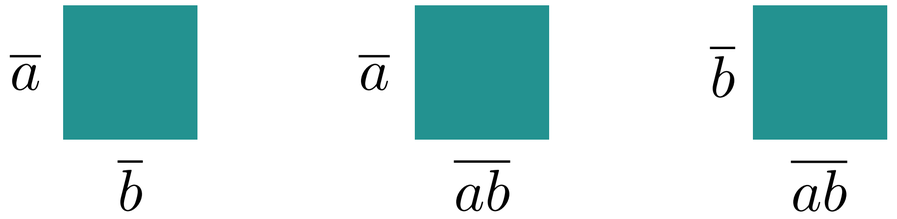
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

In a $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold, discrete torsion $\in H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and the nontrivial element acts as a sign on the twisted sectors



the same sectors which were omitted above.

$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Adding the universes projects out some sectors — interference effect.

Example, cont'd

Compute the partition function of $[X/D_4]$

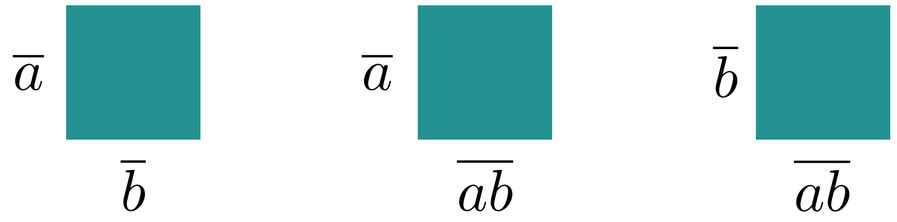
(T Pantev, ES '05)

$$Z_{T^2}\left([X/D_4]\right) = \frac{|\mathbb{Z}_2\times\mathbb{Z}_2|}{|D_4|}|\mathbb{Z}_2|^2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

$$= 2\left(Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]\right) - \left(\text{some twisted sectors}\right)\right)$$

Discrete torsion is $H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$,

and acts as a sign on the twisted sectors



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$$Z_{T^2}\left([X/D_4]\right) = Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) + Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

$$\text{QFT}\left([X/D_4]\right) = \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \coprod \text{QFT}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Example, cont'd

$$Z_{T^2}\left([X/D_4]\right) \;=\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{w/o d.t.}}\right) \;+\; Z_{T^2}\left([X/\mathbb{Z}_2\times\mathbb{Z}_2]_{\text{d.t.}}\right)$$

Matches prediction of decomposition

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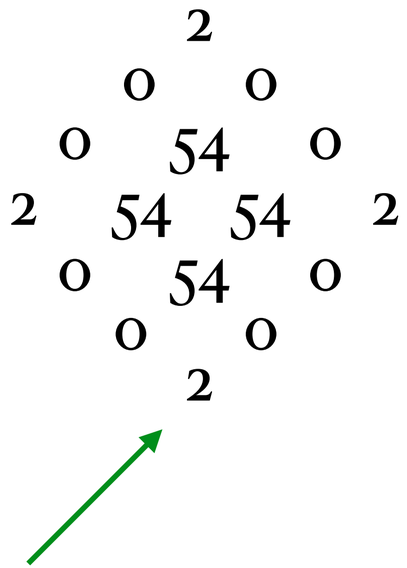
The computation above demonstrated that the partition function on T^2
has the form predicted by decomposition.
The same is also true of partition functions at higher genus
— just more combinatorics.
(see hep-th/0606034, section 5.2 for details)

Only slightly novel aspect: in gen'l, one finds dilaton shifts,
which mostly I'll suppress in this talk.

Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$



Signals mult' components / cluster decomp' violation

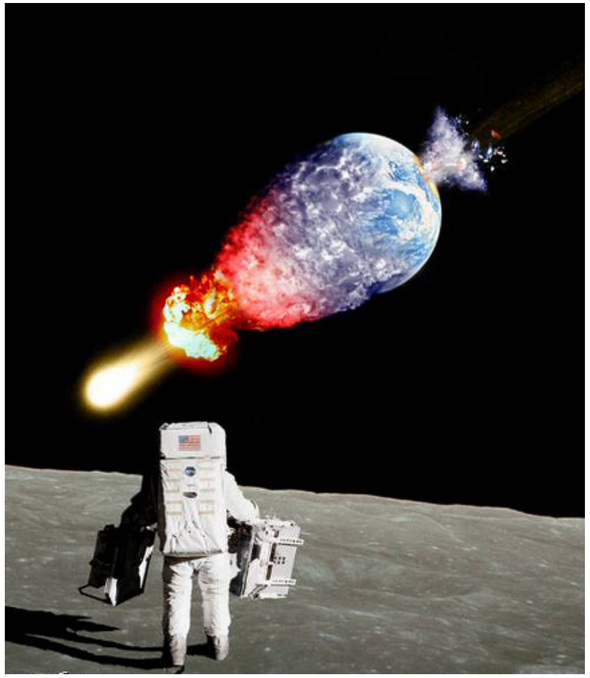
If we didn't know about decomposition, the 2's in the corners would be a problem...

A big problem!

They signal a violation of cluster decomposition, the same axiom that's violated by restricting instantons.

Ordinarily, I'd assume that the computation was wrong.

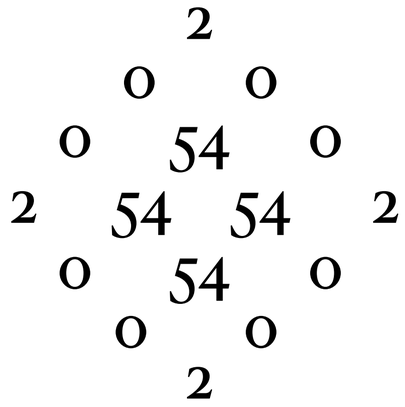
However, decomposition saves the day....



Example, cont'd

Massless states of $[X/D_4]$ for $X = T^6$ (T Pantev, ES '05)

Massless states of $[T^6/D_4]$



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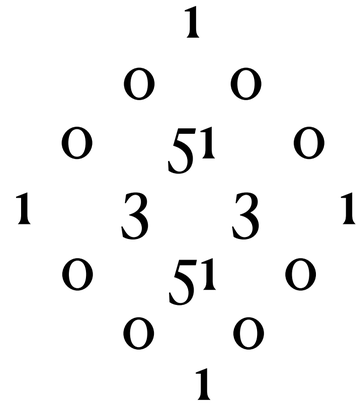
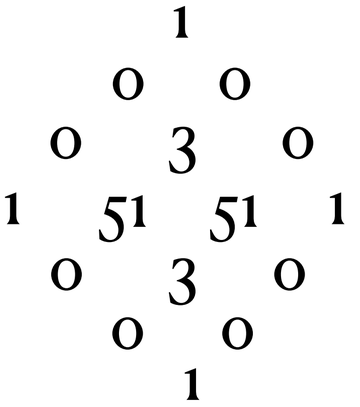


spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb'

w/o d.t.

+



spectrum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orb'

w/ d.t.

Signals mult' components / cluster decomp' violation

matching the prediction of decomposition

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{w/o\,d.t.}}\right) \coprod \mathrm{CFT}\left([X/\mathbb{Z}_2 \times \mathbb{Z}_2]_{\mathrm{d.t.}}\right)$$

This computation was not a one-off, but in fact verifies a prediction in Hellerman et al '06 regarding QFTs in (1+1)-dims with 1-form symmetry.

Another example: Triv'ly acting subgroup **not** in center

Consider $[X/\mathbb{H}]$, $\mathbb{H}$ = eight-element gp of unit quaternions,

where $\langle i \rangle = \mathbb{Z}_4 \subset \mathbb{H}$ acts trivially.

Decomposition predicts

$$\text{QFT}\left([X/\Gamma]\right) \;=\; \text{QFT}\left(\left[\frac{X \times \hat{K}}{G}\right]_{\hat{\omega}}\right)$$

(Hellerman et al '06)

where $\hat{K}$ = irreps of K

$\hat{\omega}$ = discrete torsion on universes

Here, $G = \mathbb{H}/\langle i \rangle = \mathbb{Z}_2$ acts nontriv'ly on $\hat{K} = \mathbb{Z}_4$, interchanging 2 elements,

so $$\text{QFT}\left([X/\mathbb{H}]\right) \;=\; \text{QFT}\left(X \coprod [X/\mathbb{Z}_2] \coprod [X/\mathbb{Z}_2]\right)$$

(Hellerman et al, hep-th/0606034, sect. 5.4)

— different universes; $X \neq [X/\mathbb{Z}_2]$

— easily checked

Quick note: applications of decomposition in 2d orbifolds

One recent application (Robbins et al '21) was to understand Wang-Wen-Witten's work on anomaly resolution (WWW '17).

Briefly, given an orbifold $[X/G]$ with a gauge anomaly,
Wang-Wen-Witten abstractly construct a related orbifold $[X/\Gamma]_B$,
with a trivially-acting $K \subset \Gamma$,
which in principle is anomaly free.

However, decomposition implies (Robbins et al '21)

$$[X/\Gamma]_B = \coprod [X/\text{ anomaly-free subgp of } G]$$

which gives a simple way to understand why WWW's procedure works.

So far we've discussed orbifolds, but analogous statements hold in gauge theories.

Decomposition:

$$\text{QFT}(G-\text{gauge theory}) = \coprod_{\text{char's } \hat{K}} \text{QFT}\left(G/K-\text{gauge theory w/ discrete theta angles}\right)$$

Example: pure $SU(2)$ gauge theory = sum $SO(3)_+ + SO(3)_-$ pure gauge theories

where $\pm$ denote discrete theta angles (w_2)

$$SU(2) \text{ instantons (bundles)} \subset SO(3) \text{ instantons (bundles)}$$

The discrete theta angles weight the non-$SU(2)$ $SO(3)$ instantons so as to cancel out of the partition function of the disjoint union.

Summing over the $SO(3)$ theories projects out some instantons, giving the $SU(2)$ theory.

Restriction on nonperturbative sectors,

implemented by a sum over universes.

Before going on, let's quickly check these claims for pure $SU(2)$ Yang-Mills in 2d.

The partition function Z, on a Riemann surface of genus g, is

(Migdal, Rusakov)

$$Z(SU(2)) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps

$$Z(SO(3)_+) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SO(3) reps

(Tachikawa '13)

$$Z(SO(3)_-) = \sum_R (\dim R)^{2-2g} \exp(-AC_2(R))$$ Sum over all SU(2) reps that are not SO(3) reps

Result: $Z(SU(2)) = Z(SO(3)_+) + Z(SO(3)_-)$ as expected.

Aside: for pure 2d YM, there exists a more extreme decomposition to invertible field theories. (Nguyen, Tanizaki, Unsal '21)

Aside: a common feature of these theories:
violation of cluster decomposition

As Weinberg taught us years ago,
restricting instantons violates cluster decomposition,
and as we have seen,
instanton restriction is a common feature in these theories.

Now, a disjoint union of QFTs also violates cluster decomposition,
but in a trivially controllable fashion.

Lesson: restricting instantons OK,
so long as one has a disjoint union.

(Hellerman, Henriques, T Pantev, ES, M Ando, hep-th/0606034)

Mathematical interpretation:

So far I've just talked abstractly about 2d QFTs & 1-form symmetries.

This has a mathematical interpretation: "gerbes"

A G-gerbe is a fiber bundle

whose fibers are 1-form symmetry `groups', specifically, BG.

$\neq$

A sigma model whose target is a G-gerbe has a global BG symmetry,

just as a sigma model whose target is a G-bundle has a global G-symmetry,

from translations on the fibers.

Furthermore, $BG = [\text{point}/G]$

so whenever a group acts trivially,

you should expect a gerbe structure (1-form symmetry) somewhere.

Mathematical interpretation:

Twenty years ago, I was interested in studying
`sigma models on gerbes' as possible sources of new string compactifications.

Potential issues, since solved:

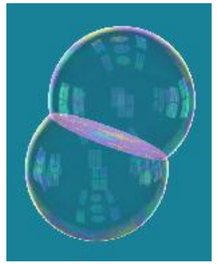
construction of QFT; cluster decomposition; moduli;
mod' invariance & unitarity in orbifolds; potential presentation-dependence.

What we eventually learned was that these theories are well-defined,
but,
are disjoint unions of ordinary theories, at least in (2,2) susy cases,
because of decomposition.

Not really new compactifications, but instead other applications.
I'll list some of my favorites next....

Application: GW invariants

The Gromov-Witten (GW) invariants count minimal-area surfaces in a given space.



There exists a def'n of GW invariants of gerbes.
(Chen, Ruan; Abramovitch, Graber, Vistoli ~2000)

Decomposition predicts,
GW invariants of a gerbe = sum of GW invariants of universes

Checked by (H-H Tseng, Y Jiang, et al '08 on)

Application: GLSMs

(Caldararu et al '07)

Consider the GLSM for e.g. $\mathbb{P}^3[2,2] = T^2$.

This is a $U(1)$ gauge theory, with ϕ_i charge $+1$, p_a charge -2.

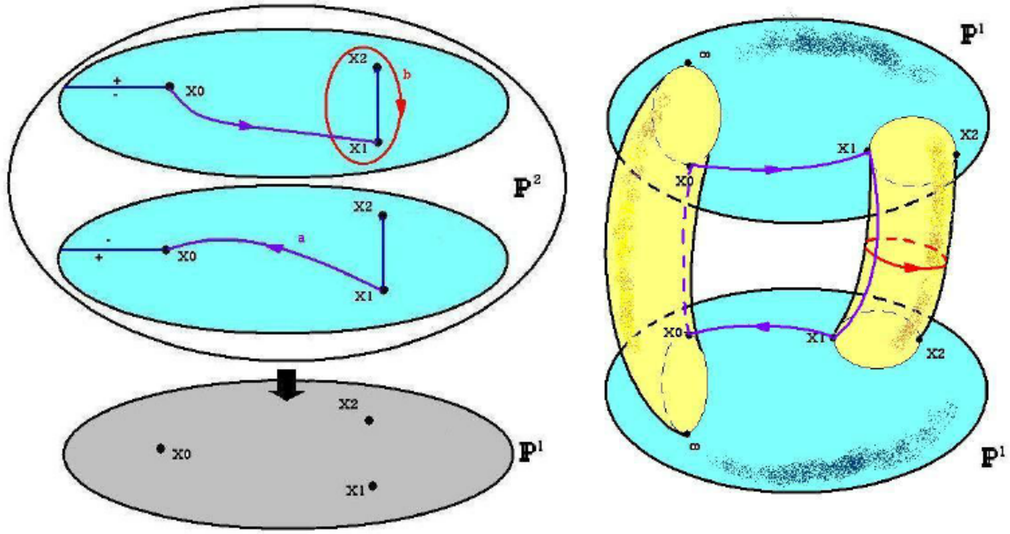
The LG point has superpotential

$$W = \sum_{ij} A^{ij}(p)\,\phi_i\phi_j \qquad \text{— mass matrix for } \phi \text{ fields.}$$

Away from zeroes of eigenvalues of A^{ij},

looks like sigma model on $\mathbb{P}^1 = \text{Proj}\,\mathbb{C}[p_1, p_2]$, with $B\mathbb{Z}_2$ symmetry.

Decomposition $\Rightarrow$ Double cover of $\mathbb{P}^1$, branched over $\{\det A = 0\} = \{4 \text{ points}\}$



Another T^2 !
geometry
realized
nonperturbatively
via decomposition

Application: elliptic genera of pure susy gauge theories

(R Eager, ES '20)

We can use decomposition to predict elliptic genera of pure (2,2) susy gauge theories, using knowledge of IR susy breaking for various discrete theta angles.

Example: for $SU(k)/\mathbb{Z}_k$, susy unbroken only for discrete theta $\theta = -(1/2)k(k-1) \mod k$

(as derived from 2d nonabelian mirrors)

$$\mathrm{EG}(G/K,\theta) = 0 \quad \text{if susy broken in IR}$$

$$\textbf{Decomposition} \Rightarrow \mathrm{EG}(G) = \sum_\theta \mathrm{EG}(G/K,\theta)$$

Can then algebraically recover elliptic genera.

$$\text{Example:} \quad \mathrm{EG}(SU(k)/\mathbb{Z}_k,\theta) = (1/k)\mathrm{EG}(SU(k)) \sum_{m=0}^{k-1} (-)^{m(k+1)} \exp(im\theta)$$

For $k = 2$, matches (Kim, Kim, Park '17).

Numerous other low-rank exs checked with susy localization.

Goal for today: a (hopefully pedagogical) introduction to decomposition

Outline:

- Decomposition in 2d orbifolds, from a perspective that will motivate later cases

  Global 1-form symmetry from gauging trivially-acting 0-form symmetry

  Aside on gauge theory examples

- **Decomposition in 3d orbifolds**

  Global 2-form symmetry from gauging trivially-acting 1-form symmetry

- Decomposition in 3d Chern-Simons

  Global 2-form symmetry from gauging trivially-acting 1-form symmetry

Three-dimensional examples

Let's construct an example of a decomposition in 3d.

We need a theory with a global 2-form symmetry.

One way to get that is by gauging a trivially-acting one-form symmetry,
by which we mean, for example, line operators have no braiding.

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad\qquad [\omega] \in H^3(G, K)$$

G, K finite; K abelian; BK acts trivially.

Since BK acts trivially, this theory should have a global 2-form symmetry, & so decompose.

Let's see that explicitly.

Projectors: Projectors are constructed from monopole operators associated to the BK, which generate K-gerbes on surrounding S^2's.

For example, if $K = \mathbb{Z}_k$, then as $\mathbb{Z}_k$-gerbes on S^2 have one generator, there is one generating monopole operator, call it $\hat{z}$, with the property $\hat{z}^k = 1$.

$$\Pi_n = \frac{1}{k} \sum_{m=0}^{k-1} \xi^{mn} \hat{z}^m \qquad \text{where } \xi = \exp(2\pi i/k)$$

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

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Since BK acts trivially, this theory should have a global 2-form symmetry, & so decompose.

We find:

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho \circ \epsilon}\right)$$

(closely analogous to 2d orbifolds with trivially-acting K)

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad\qquad [\omega] \in H^3(G, K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function:

In general terms, the path integral for the orbifold $[X/\Gamma]$ involves a sum over

- principal Γ-bundles E over the 3-manifold M_3
- Maps $E \to X$

just like an ordinary orbifold.

Also, since BK acts trivially, the twisted sectors will be those of a G orbifold.

However, those G-twisted sectors are restricted....

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad\qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function:

On T^3, the sum over Γ-twisted sectors maps to a sum over G-twisted sectors such that

$$\epsilon(g_1,g_2,g_3) = \frac{\omega(g_1,g_2,g_3)}{\omega(g_2,g_1,g_3)}\frac{\omega(g_3,g_1,g_2)}{\omega(g_1,g_3,g_2)}\frac{\omega(g_2,g_3,g_1)}{\omega(g_3,g_2,g_1)} = 1 \in K$$

— restriction on nonperturbative sectors

We can implement that restriction by inserting a delta function

$$\delta(\epsilon - 1) = \frac{1}{|K|}\sum_{\rho\in\hat{K}} \rho\circ\epsilon$$

Partition function....

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad\qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function on T^3:

Delta f'n, enforcing constraint

$$\begin{aligned} Z_{T^3}([X/\Gamma]) &= \frac{|H^0(T^3,K)|}{|H^1(T^3,K)|}\frac{1}{|H^0(T^3,G)|}\sum_{z_{1-3}\in K}\sum_{g_{1-3}\in G}\delta(\epsilon-1)\,Z(g_1,g_2,g_3) \\ &= \frac{1}{|K|^2|G|}|K|^3\sum_{g_{1-3}\in G}\frac{1}{|K|}\sum_{\rho\in\hat{K}}(\rho\circ\epsilon)(g_1,g_2,g_3)\,Z(g_1,g_2,g_3) \\ &= \sum_{\rho\in\hat{K}} Z_{T^3}\left([X/G]_{\rho\circ\epsilon}\right) \end{aligned}$$

where $\rho\circ\epsilon$ defines C-field-analogue of discrete torsion

Adding the universes projects out some sectors — interference effect.

Three-dimensional examples

(T Pantev, D Robbins, T Vandermeulen, ES 2204.13708)

Example: Consider an orbifold $[X/\Gamma]$ where

$$1 \longrightarrow BK \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \qquad [\omega] \in H^3(G,K)$$

G, K finite; K abelian; BK acts trivially. Claim $[X/\Gamma]$ decomposes.

Partition function on T^3:

$$Z_{T^3}([X/\Gamma]) = \sum_{\rho\in\hat{K}} Z_{T^3}\left([X/G]_{\rho\circ\epsilon}\right)$$

where $\rho \circ \epsilon$ defines C-field-analogue of discrete torsion

consistent with

$$\mathrm{QFT}([X/\Gamma]) = \coprod_{\rho\in\hat{K}} \mathrm{QFT}\left([X/G]_{\rho\circ\epsilon}\right)$$

Decomposition

Similar results arise on other 3-manifolds.

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- **Decomposition in 3d Chern-Simons**

  Global 2-form symmetry from gauging trivially-acting 1-form symmetry

Three-dimensional examples

(T Pantev, ES, 2206.14824)

Example: Chern-Simons theories

Chern-Simons theories are particularly interesting for these ideas.

For example, classically AdS_3 is Chern-Simons for $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$,
so understanding decomposition in Chern-Simons theories
may give toy models of issues in gravity theories
such as Marolf-Maxfield factorization.

So, what's the decomposition in Chern-Simons ?

Three-dimensional examples

(T Pantev, ES, 2206.14824)

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

There is an associated `crossed module'

$$1 \longrightarrow K(=\ker d) \longrightarrow A \xrightarrow{d} H \longrightarrow G(=H/\mathrm{im}\, d) \longrightarrow 1$$

Similar remarks apply: only restricted G bundles can appear.

To implement that restriction, must sum over universes....

Conjecture:

$$\text{Chern-Simons}(H)\,/\,BA \;=\; \coprod_{\rho\in\hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

Decomposition

Three-dimensional examples

(T Pantev, ES, 2206.14824)

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

Conjecture: $$\text{Chern-Simons}(H)\,/\,BA \;=\; \coprod_{\rho\in\hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

Example: Chern-Simons($SU(2)$) / $B\mathbb{Z}_2$ where the $B\mathbb{Z}_2$ acts via the center

$$1 \longrightarrow K\,(=1) \longrightarrow \mathbb{Z}_2 \xrightarrow{d} SU(2) \longrightarrow SO(3)\,(=SU(2)/\mathrm{im}\,d) \longrightarrow 1$$

so predict

$$\text{Chern-Simons}(SU(2))\,/\,B\mathbb{Z}_2 \;=\; \text{Chern-Simons}(SO(3))$$

which is a standard result.

Three-dimensional examples

(T Pantev, ES, 2206.14824)

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

Conjecture: $$\text{Chern-Simons}(H)\,/\,BA \;=\; \coprod_{\rho\in\hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

Example: Chern-Simons($SU(2)$) / $B\mathbb{Z}_4$ where the $B\mathbb{Z}_4$ maps to the center

$$1 \longrightarrow K\,(=\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4 \xrightarrow{d} SU(2) \longrightarrow SO(3)\,(=SU(2)/\mathrm{im}\,d) \longrightarrow 1$$

so predict

$$\text{Chern-Simons}(SU(2))\,/\,B\mathbb{Z}_4 \;=\; \coprod_{\rho\in\hat{\mathbb{Z}}_2} \text{Chern-Simons}(SO(3))_{\omega(\rho)}$$

where here ω couples to third Stiefel-Whitney class.

Three-dimensional examples

(T Pantev, ES, 2206.14824)

Example: Chern-Simons theories

Consider Chern-Simons(H) / BA for A finite & abelian.

Conjecture: $$\text{Chern-Simons}(H)\,/\,BA \;=\; \coprod_{\rho\in\hat{K}} \text{Chern-Simons}(G)_{\omega(\rho)}$$

How to check?

For example, boundaries. Above becomes

$$\text{WZW}(H)/A \;=\; \coprod_{\rho\in\hat{K}} \text{WZW}(G)_{\theta(\rho)}$$

where the boundary discrete theta angle related to bulk via transgression.

Can show, in fact, boundary discrete theta angle = discrete torsion,
and the predicted boundary decomposition = standard 2d orbifold decomposition.

Summary

Decomposition: `one' theory = disjoint union of several

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Thank you for your time!