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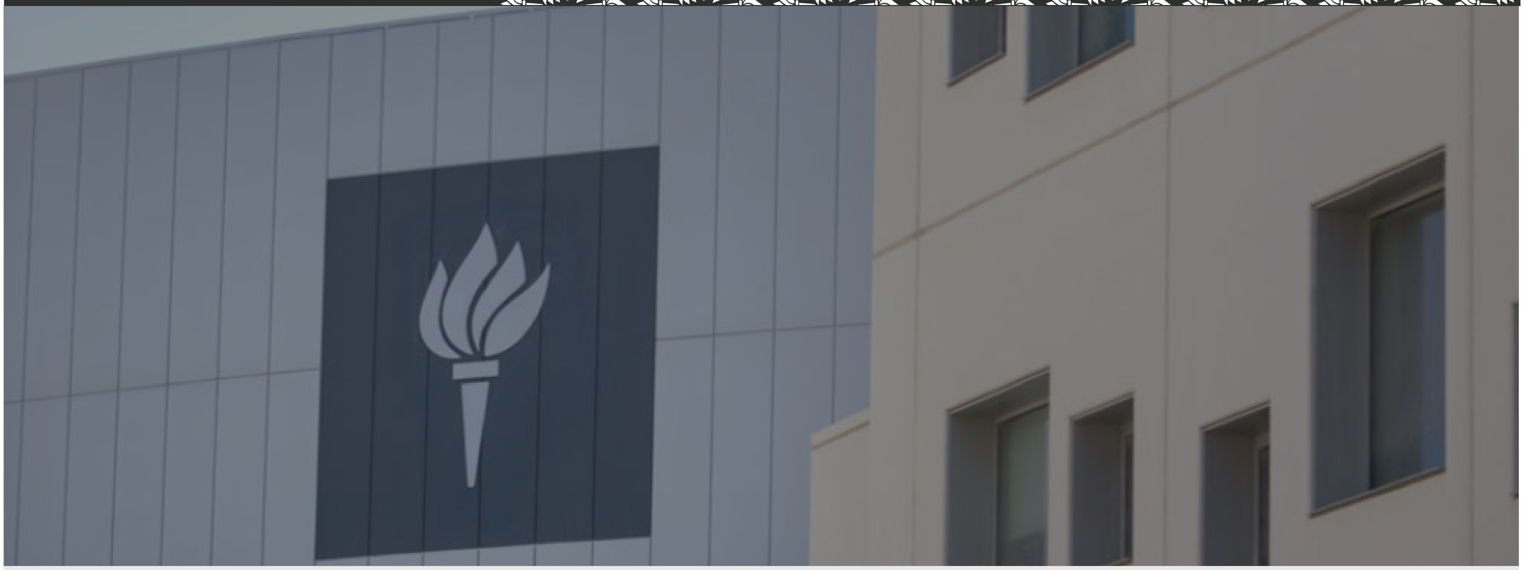

Dual Auctions for Assigning Winners and Compensating Losers

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Dual Auctions for Assigning Winners and Compensating Losers

Matt Van Essen* John Wooders†

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Abstract

We study the problem of allocating goods (or rights) and chores when participants have equal claim on a unit of the good or equal obligation to undertake a chore. We propose two dynamic auctions for solving problems of this type: a "goods" auction and a "chore" auction, which are duals of one another. Either auction can be used for allocating goods or chores by suitably defining a good or a chore. The auctions are efficient and payoff equivalent. We provide necessary and sufficient conditions for equilibrium for general utility functions for both auctions, and provide closed-form solutions when bidders are risk neutral and when they are CARA risk averse. The auctions have the same limit equilibrium bid function as bidders become infinitely risk averse. We show that the limit bid function is also the unique maxmin perfect strategy for both auctions.

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1 Introduction

We study the problem of allocating K identical units of a good, or K identical chores, to N individuals, where $K < N$ and each individual has an equal claim to a unit of the good or an equal obligation to undertake a chore. Problems of this kind arise frequently. For example, N individuals may have the right to fish but, to reduce overfishing, the rights of $N - K$ are to be withdrawn. The problem then is to allocate the remaining fishing rights to the K individuals who value them most highly, with those individuals who retain the right compensating those who forfeit it. Likewise, if there are K identical chores that must be completed, the problem is to assign chores to the K individuals who have the lowest cost for undertaking them, with the $N - K$ individuals excused from the chore compensating the rest.

The dynamic auctions we propose for allocating goods (or rights) and assigning chores have a number of desirable properties: They are simple. They treat the participants, henceforth bidders, symmetrically. Whether bidders are risk neutral or risk averse, goods are allocated in equilibrium to the bidders with the highest values for consuming them and chores are assigned to the bidders with the lowest cost for undertaking them. Participation is individually rational: a bidder who participates obtains an equilibrium payoff that exceeds the payoff he would obtain were the goods or chores assigned randomly.

We first describe the chore auction for allocating K chores to N bidders: The auction takes place over $N - K$ rounds. At each round, the price, starting from the highest possible cost, descends continuously. A bidder may drop out at any point. A bidder who drops out is excused from undertaking the chore, but must pay compensation equal to the price at which he drops out, with the compensation to be shared equally among the K bidders who eventually undertake a chore. Each time a bidder drops out, the price is reset to the highest possible cost and the process repeats. The auction ends when

$N - K$ bidders have dropped out. The K bidders remaining undertake the chore and each receives $1/K$-th of the total compensation promised by the excused bidders.

The structure of the goods auction that assigns K goods to N bidders is the same, except that at each round the price starts at zero and ascends. A bidder who drops out surrenders his claim to a unit of the good and in return is promised compensation equal to his dropout price from the eventual winners of a unit of the good. The auction ends when $N - K$ bidders have dropped out. The K bidders remaining each obtain a unit of the good and pay $1/K$-th of the total compensation promised to the bidders who surrendered their claim.

For both auctions we characterize necessary and sufficient conditions for bid functions to form a symmetric equilibrium in increasing and differentiable strategies when bidders have independent private costs or values. We provide closed-form solutions for the (unique) symmetric equilibrium when bidders are risk neutral and when they are CARA risk averse. Since the auctions are dynamic, an equilibrium is characterized by a sequence of bid functions, where the t-th bid function identifies a bidder's dropout price in round t as a function of his cost (or value) and the prices at which bidders have dropped out in prior rounds.

In both auctions, bidders drop out earlier as they become more risk averse. In the chore auction this means bidders pay more compensation to be excused from the chore, whereas in the goods auction they accept less compensation in return for surrendering their claim to the good. For fixed K and N, as bidders become infinitely risk averse the equilibrium bidding strategies of the auctions converge to the same limit bidding strategy, which is linear in costs or values.

In the actual application of any allocation mechanism, the participants may be concerned with their worst-case outcome. For each auction, we identify a bidder's maxmin payoff as the maximum payoff that he can guarantee himself at the outset of the auction, regardless of the costs (or values) and strategies of the other bidders. We show that the maxmin payoff in the chore auction of a bidder whose cost x is $-Kx/N$, and the maxmin payoff in the goods auction for a bidder whose value is x is Kx/N.

A maxmin strategy is a strategy that guarantees a bidder at least his maxmin payoff. There are many such strategies in both auctions, and we focus on a natural refinement. A maxmin strategy is "perfect" if it maximizes the payoff that a bidder can guarantee himself starting from any history of play. We show that there is a unique maxmin perfect strategy for both the chore and the goods auctions. Furthermore, the maxmin perfect strategy is the same for both auctions and it coincides with the (symmetric) equilibrium bid strategy as bidders become infinitely risk averse.

Clearly, the chore auction and goods auction are closely related. Either auction can be used whether the problem is to allocate chores or goods. Consider, for example, the problem of assigning K goods to N bidders. It can be solved via a goods auction with $N - K$ rounds. Alternatively, we can assign a good to each of the N bidders and define the chore to be the surrender of the good. The allocation problem can then be solved by a chore auction with K rounds, where each bidder who drops pays compensation to be excused from the chore (i.e., to keep their good). The two auctions are not equivalent: in the first, $N - K$ bidders receive compensation for surrendering their claim to the good, while in the second K bidders pay compensation to keep their good. We show, nonetheless that the two auctions are payoff equivalent when bidders are risk neutral. Indeed, we provide a general payoff equivalence theorem for all symmetric, efficient, and budget balanced mechanisms.

Although payoff equivalent, there may be practical reasons to use one auction over the other. For instance, if there is a single unit of the good, then the goods auction ends after $N-1$ rounds, whereas the chore auction ends after only one round.

Related Literature

The problem of assigning a single good, i.e., $K=1$, to one of N players with equal claims is the well-known problem of dissolving a partnership. The most commonly used mechanism for dissolving two-person partnerships is the "Texas-Shootout," where one partner proposes a price and the other partner is compelled to either buy his partner's share or sell his own share at that price. (This is simply the classic "divide and choose" procedure, studied in the cake cutting literature, where the indivisible good is made divisible by using money transfers.)[1] de Frutos and Kittsteiner (2008) study the Texas-Shootout when the proposer is determined endogenously via an auction. McAfee (1992) and de Frutos (2000) characterize equilibrium bidding for the Winner's bid and the Loser's bid auction for two-person partnerships. Wasser (2013) studies a family of auctions for dissolving partnerships in which the distributions of the bidders' values and the ownership shares are asymmetric. Morgan (2000) and Brooks, Landeo, and Spier (2010) study dissolving partnerships in common value settings.

For general partnerships, i.e., $N \geq 2$ and $K=1$, Cramton, Gibbons, and Klemperer (1987) provide necessary and sufficient conditions for a partnership to be efficiently dissolvable and, when it is, they identify a (static) auction that dissolves it. They show that equal share partnerships are always dissolvable by simple auctions. Van Essen and Wooders (2016) propose a dynamic auction for dissolving such partnerships and characterize equilibrium

[1] There are many connections between the literature on dissolving partnerships and the cake cutting literature. Classic papers on cake cutting include Steinhaus (1948) and Dubins and Spanier (1961). Brams and Taylor (1996) surveys this literature. Su (1999) discusses how an envy free cake-cutting algorithm can be applied to dividing a chore or sharing a cost.

bidding when bidders are risk neutral or CARA risk averse. Only McAfee (1992) and Van Essen and Wooders (2016) allow for bidder risk aversion.

The goods auction studied here can be viewed as a mechanism for reorganizations of partnerships that reduce the number of partners from N to K, for any $K < N$. In this case, $N - K$ partners must be compensated for surrendering their share of the partnership. Our results show that the goods auction efficiently reorganizes partnerships when bidders are either risk neutral or risk averse. Alternatively, the chore auction can be applied instead and, when K is small, it reorganizes a partnership with an auction that takes fewer rounds.

In the papers above, the solution concept is Bayes Nash equilibrium. If the solution concept is dominant strategy incentive compatibility, it is well known from Green and Laffont (1977) and Walker (1980) that there is no mechanism for our setting that is efficient and budget balanced. Long, Mishra, and Sharma (2017) relaxes efficiency and provides a mechanism for the $K = 1$ problem that is dominant strategy incentive compatible, budget balanced, but not efficient. It is nearly efficient when the number of bidders is large. Long (2016) extends this mechanism to the general K case.

Finally, our paper contributes to a literature on multi-unit sequential auctions with single-unit demands and bidder risk aversion. Recent contributions include Mezzetti (2011) and Hu and Zou (2015) who provide conditions for the sequence of prices received by a seller to be increasing or decreasing. In our setting we study the allocation of homogenous chores and goods when the bidders have equal obligations or claims. Of course, in our context there is no seller.

2 The Model

There are N bidders and $K < N$ identical chores or $K < N$ identical goods. The bidders' costs and values for the chore and the good, respectively, are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Bidders have a common utility function u, where $u' > 0$ and $u'' \leq 0$.

Let $X_1, \ldots, X_N$ be N independent draws from F. When the X_i's are costs, it is convenient to order them from highest to lowest. Let $Y_1^{(N)}, \ldots, Y_N^{(N)}$ be a rearrangement of the X_i's such that $Y_1^{(N)} \geq Y_2^{(N)} \geq \ldots \geq Y_N^{(N)}$. The joint density of $Y_1^{(N)}, \ldots, Y_N^{(N)}$ is

$$g_{1,\ldots,N}^{(N)}(y_1, \ldots, y_N) = N! \prod\nolimits_{i=1}^{N} f(y_i)$$

if $y_1 \geq y_2 \geq \ldots \geq y_N$ and zero otherwise. The conditional density of $Y_t^{(N)}$ given $Y_1^{(N)} = y_1, \ldots, Y_{t-1}^{(N)} = y_{t-1}$ is

$$g_t^{(N)}(y_t | y_1, \ldots, y_{t-1}) = g_t^{(N)}(y_t | y_{t-1}) = (N - (t-1)) f(y_t) \frac{F(y_t)^{N-t}}{F(y_{t-1})^{N-(t-1)}}$$

if $y_1 \geq \ldots \geq y_{t-1}$ and is zero otherwise.

When the X_i's are values, conversely, it is convenient to order them from lowest to highest. Let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$. The joint of $Z_1^{(N)}, \ldots, Z_N^{(N)}$ is

$$h_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = N! \prod\nolimits_{i=1}^{N} f(z_i)$$

if $z_1 \leq z_2 \leq \ldots \leq z_N$ and zero otherwise. The conditional density of $Z_t^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_{t-1}^{(N)} = z_{t-1}$ is

$$h_t^{(N)}(z_t | z_1, \ldots, z_{t-1}) = h_t^{(N)}(z_t | z_{t-1}) = (N - (t-1)) f(z_t) \frac{[1 - F(z_t)]^{N-t}}{[1 - F(z_{t-1})]^{N-(t-1)}}$$

if $0 \leq z_1 \leq \ldots \leq z_{t+1}$ and is zero otherwise.

The Chore Auction

The chore auction selects K of N bidders to undertake K identical chores or, equivalently, it selects K bidders to undertake a single chore that requires K bidders to complete. At each round there is a descending clock auction in which the price starts at $\bar{x}$ and decreases continuously. Bidders may drop out at any point. A bidder who at round t drops out at p_t is excused from the chore, pays p_t in compensation, and obtains a payoff of $u(-p_t)$. A new round then begins and this process repeats until exactly K bidders remain (i.e., for $N-K$ rounds). Each of these bidders undertakes a chore and receives an equal share of the total compensation, i.e., $\frac{1}{K}\Sigma_{j=1}^{N-K}p_j$, paid by the bidders who dropped. The payoff of a bidder with cost x who undertakes a chore is $u(\frac{1}{K}\Sigma_{j=1}^{N-K}p_j - x)$.

The Goods Auction

The goods auction selects K of N bidders to receive one of K identical items. At each round there is an ascending clock auction in which the price starts at zero and increases continuously. Bidders may drop out at any point. A bidder who at round t drops out at p_t surrenders his claim to an item, receives p_t in compensation, and obtains a payoff of $u(p_t)$. A new round then begins and this process repeats until exactly K bidders remain (i.e., for $N-K$ rounds). Each of these bidders receives an item and pays an equal share of the total compensation, i.e., $\frac{1}{K}\Sigma_{j=1}^{N-K}p_j$, promised to the bidders who dropped. The payoff of a bidder with value x who receives an item is $u(x - \frac{1}{K}\Sigma_{j=1}^{N-K}p_j)$.

For both auctions, a strategy for bidder is a list of $N-K$ functions which identifies a bidder's drop out price at each round of the auction. We write $\mathbf{p}_t$ for $(p_1, \ldots, p_t)$ and take $\mathbf{p}_0 = 0$. For the chore auction we denote a strategy by $\delta = (\delta_1, ..., \delta_{N-K})$, where $\delta_t(x; \mathbf{p}_{t-1})$ gives the dropout price in the t-th round of a bidder with cost x when $t-1$ bidders have dropped out at prices $\mathbf{p}_{t-1}$. Likewise, for the goods auction we denote a strategy by $\beta = \left(\beta_1, ..., \beta_{N-K}\right)$, where $\beta_t(x; \mathbf{p}_{t-1})$ gives the dropout price in the t-th round of a bidder with value x when $t-1$ bidders have dropped out at prices $\mathbf{p}_{t-1}$.

3 Equilibrium

Proposition 1(i) identifies necessary conditions for δ to be a symmetric equilibrium of the chore auction in increasing and differentiable strategies. Proposition 1(ii) establishes that the necessary conditions are also sufficient. The analogous result for the goods auction is provided in the Appendix as Proposition 1′.

Proposition 1: *(i) Any symmetric equilibrium δ of the chore auction in increasing and differentiable bid strategies satisfies the following system of differential equations:*

$$
\begin{aligned}
& u'(-\delta_{N-K}(x; \mathbf{p}_{N-K-1}))\delta'_{N-K}(x; \mathbf{p}_{N-K-1}) \\
= & \left[u(-\delta_{N-K}(x; \mathbf{p}_{N-K-1})) - u\left(\frac{1}{K} \left[\delta_{N-K}(x; \mathbf{p}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i \right] - x \right) \right] \bar{\lambda}^N_{N-K}(x),
\end{aligned}
$$

and, for $t \in \{1, \ldots, N-K-1\}$, that

$$
\begin{aligned}
& u'\left(-\delta_t(x; \mathbf{p}_{t-1})\right) \delta'_t(x; \mathbf{p}_{t-1}) \\
= & \left[u\left(-\delta_t(x; \mathbf{p}_{t-1})\right) - u(-\delta_{t+1}(x; \mathbf{p}_{t-1}, \delta_t(x; \mathbf{p}_{t-1})))\right]\bar{\lambda}^N_t(x),
\end{aligned}
$$

where

$$
\bar{\lambda}^N_t(x) = (N-t)\frac{f(x)}{F(x)}.
$$

(ii) If $\delta = (\delta_1, \ldots, \delta_{N-K})$ is a solution to the system of differential equations in (i), then it is an equilibrium.

These differential equations have a simple interpretation. At any round, the marginal benefit to a bidder from remaining in the chore auction a moment longer is that he pays less compensation if no other bidder drops out in the interim. At the last round, the marginal cost of remaining a moment longer is that a rival bidder drops out and the bidder must undertake the chore. At earlier rounds, the marginal cost of remaining is that a rival bidder drops out, in which case the bidder continues into the next round and pays the compensation that his rival would have paid. The differential equations state that, at each round, marginal benefit and cost are equalized at equilibrium.

We will provide closed-form expressions for the unique symmetric equilibrium when the bidders are either risk neutral or CARA risk averse. When bidders have index of risk aversion α, we denote the equilibrium bid functions for the chore and the goods auction by δ_t^{α} and β_t^{α}, respectively. Bidders are risk neutral when $\alpha = 0$.

3.1 Risk Neutral Bidders

Proposition 2 characterizes the equilibrium bid functions in the chore and the goods auctions when bidders are risk neutral.

Proposition 2: *Suppose that bidders are risk neutral.*
(P 2.1) *The unique symmetric equilibrium in increasing and differentiable strategies for the chore auction is, for* $t = 1, \ldots, N-K$,

$$\delta_t^0(x; \mathbf{p}_{t-1}) = \frac{K}{N-t+1}\left(E\left[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] - \frac{1}{K}\sum_{i=1}^{t-1} p_i\right).$$

(P 2.2) *The unique symmetric equilibrium in increasing and differentiable strategies for the goods auction is, for* $t = 1, \ldots, N-K$,

$$\beta_t^0(x; \mathbf{p}_{t-1}) = \frac{K}{N-t+1}\left(E\left[Z_{N-K}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] - \frac{1}{K}\sum_{i=1}^{t-1} p_i\right).$$

The goods auction is the “dual” of the chore auction: it is an ascending bid auction rather than a descending bid auction, and bidders receive compensation to surrender their claim to the good rather than pay compensation to be excused from the chore. The dual nature of auctions is apparent from the equilibrium bid functions which have the same structure in both auctions, when these functions are expressed using the highest and lowest order statistics for the chore and goods auctions, respectively. In the bid function for the chore auction, the term $E[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}]$ is the expected cost of the lowest cost bidder that will be excused from the chore, conditional on the bidder's own cost being between the $t-1$-st and t-th highest cost. Likewise for the goods auction.

The bid functions are increasing in x for both auctions. Since the chore auction is a descending clock auction, in equilibrium the bidders with the $N-K$ highest costs drop and pay compensation to the bidders with the K lowest costs, who each undertake a chore. Since the goods auction is an ascending clock auction, in equilibrium the bidders with the $N-K$ lowest values drop and receive compensation from the bidders with the K highest values, who each receive an item.

Example 1: Suppose $N = 4$, $K = 2$, bidders are risk neutral, and values are distributed $U[0,1]$. In the chore auction, equilibrium drop prices in round 1 are

$$\delta_1^0(x) = \frac{3}{10}x$$

and in round 2 are

$$\delta_2^0(x; 1/10) = \frac{1}{2}x - \frac{1}{3}p_1.$$

Figure 1 below shows these bid functions. The round 2 bid function is shown when the realized round 1 compensation is $p_1 = 1/10$, which reveals the highest cost of a bidder is 1/3.

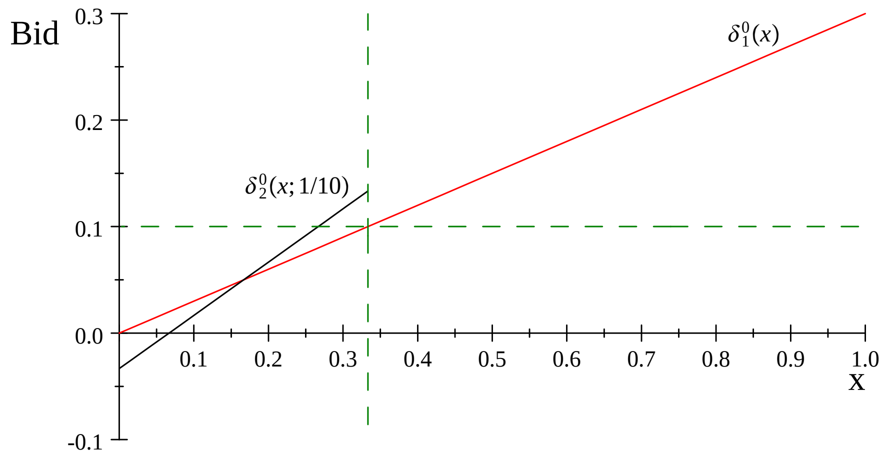

Fig. 1: Equilibrium bids by round, for $N = 4$, $K = 2$, and $U[0, 1]$.

The figure illustrates several interesting features of equilibrium. First, bid functions "jump up" whenever a bidder drops out. In this example, a bidder with cost 1/3 drops at the first round at the price $\delta_1^0(1/3) = 1/10$. At the next round, a bidder with the same cost drops at a higher price of $\delta_2^0(1/3; 1/10) = 2/15$. This is a general feature of equilibrium: more precisely $\delta_2^0(x; \delta_1^0(x)) \geq \delta_1^0(x)$ for any x. To see this, observe that if $\delta_2^0(x; \delta_1^0(x)) < \delta_1^0(x)$ then it would not be optimal for a bidder with value x to obey equilibrium and drop at round 1 at $\delta_1^0(x)$ since by waiting for a moment longer either (i) he drops at round 1 and pays lower compensation, or (ii) a rival bidder drops in the interim at some $x' < x$. In the later case, the bidder, by bidding at round 2 as though his own value were x', he pays $\delta_2^0(x'; \delta_1^0(x'))$. By continuity of the bidding strategies we have $\delta_2^0(x'; \delta_1^0(x')) < \delta_1^0(x')$ for x' close to x. Also, since δ_1^0 is increasing we have that $\delta_1^0(x') < \delta_1^0(x)$. Thus $\delta_2^0(x'; \delta_1^0(x')) < \delta_1^0(x)$ and so he pays less compensation in the later case as well. In either case the bidder pays less compensation, which is a contradiction.

Figure 1 also illustrates, as $\delta_2^0(x; p_1) < 0$ for some x, that a bidder may accept zero or negative compensation to undertake the chore. While this may at first seem surprising, it is intuitive given that the compensations promised by bidders who have dropped at earlier rounds may be large enough to make undertaking the chore attractive. In this example, when $p_1 = 1/10$, round 2 bids are negative for costs less than $1/15$. Suppose that the second highest cost is $1/15$. Then this bidder drops (and is excused from the chore) when compensation reaches zero. Each remaining bidder undertakes a chore, obtains compensation of $p_1/2 = 1/20$ and obtains a positive payoff when their cost is below $1/20$.

Payoff Equivalence

Since the chore and the goods auctions are both efficient and can be used to solve the same allocation problem, it is natural to question whether bidders would have a preference for one mechanism over the other. Proposition 3 shows that risk neutral bidders obtain the same expected payoff in every efficient symmetric equilibrium of any symmetric budget balanced mechanism. We state the proposition for goods rather than chores.

Proposition 3: *Suppose that bidders are risk neutral. Let ξ be a symmetric budget-balanced mechanism and let β be a symmetric equilibrium of ξ in which K units of a good are allocated to the bidders with the K highest values. The expected utility of a bidder with value x is*

$$\frac{K}{N} E\left[Z_{N-K}^{(N)}\right] + \int_0^x H_{N-K}^{(N-1)}(t)dt,$$

and is independent of ξ, where $H_{N-K}^{(N-1)}(t)$ is the probability that the $N-K$-th lowest of $N-1$ values is less than t.

The problem of allocating K chores among N bidders can be solved by either a chore auction (with $N - K$ rounds and one bidder excused from the chore at each round) or a goods auction with $N - K$ goods (with K rounds and one bidder assigned a chore at each round).[2] Since the requirements of Proposition 3 are met, i.e., the chore and goods auctions are symmetric and budget balanced, then the auctions are payoff equivalent at efficient and symmetric equilibria.

Corollary 1: *Suppose there are N bidders. The chore auction with K chores is payoff equivalent to the goods auction with $N-K$ goods.*

Nevertheless, there may be a practical reason to choose one auction over the other: when the number of chores K is large relative to N, the chore auction requires fewer rounds than the goods auction.

Example 2. To illustrate the payoff equivalence of the chore and goods auctions, consider the goods auction when $N = 2$, $K = 1$, and $x_1, x_2 \sim U[0,1]$. The equilibrium bid function is

$$\beta_1(x) = \frac{1}{3}x + \frac{1}{6}.$$

A bidder with value x receives compensation of $\beta_1(x)$ if $x < Z_1^{(1)}$, and he obtains the good and pays compensation of $\beta_1(Z_1^{(1)})$ if $x > Z_1^{(1)}$. His expected payoff is

$$\begin{aligned}\beta_1(x)\Pr(x &< Z_1^{(1)}) + \Big(x - E[\beta_1(Z_1^{(1)})|x \geq Z_1^{(1)}]\Big)\Pr(x \geq Z_1^{(1)}) \\ &= \left(\frac{1}{3}x + \frac{1}{6}\right)(1-x) + (x - \frac{x}{6} - \frac{1}{6})x \\ &= \frac{1}{2}x^2 + \frac{1}{6}.\end{aligned}$$

The good can also be allocated via a chore auction in which the chore for a bidder is to surrender his claim to the good. The equilibrium bid function is then

$$\delta_1(x) = \frac{1}{3}x.$$

[2] In the later case, the "good" is to be excused from the chore and thus a bidder who surrenders his claim to the good must undertake the chore.

Consider again the payoff of a bidder whose value is x. If $x > Y_1^{(1)}$, the bidder drops first, he is excused from the chore (i.e., he *retains* his claim to the good), he pays compensation of $\delta_1(x)$, and has a payoff of $x - \delta_1(x)$. If $x < Y_1^{(1)}$, then his rival drops first and the bidder undertakes the chore (i.e., surrenders his claim) and receives compensation of $\delta_1(Y_1^{(1)})$. His expected payoff is

$$\begin{aligned}(x-\delta_1(x))\,\Pr(x &> Y_1^{(1)}) + E[\delta_1(Y_1^{(1)})|x \leq Y_1^{(1)}]\,\Pr(x \leq Y_1^{(1)}) \\ &= \left(x - \frac{1}{3}x\right)x + \left(\frac{1}{3}\frac{1+x}{2}\right)(1-x) \\ &= \frac{2}{3}x^2 + \frac{1-x^2}{6} \\ &= \frac{1}{2}x^2 + \frac{1}{6}.\end{aligned}$$

Thus direct calculation establishes that a bidder with value x obtains the same payoff in both auctions, in this example.

3.2 Risk Averse Bidders

Proposition 4 characterizes equilibrium in the chore and goods auctions when bidders have constant absolute risk aversion (CARA), i.e., utility is given by

$$u^\alpha(x) = \frac{1-e^{-\alpha x}}{\alpha},$$

where $\alpha > 0$ is the index of risk aversion. Bidders are risk neutral in the limit as α approaches zero.

Proposition 4: *Suppose that bidders are CARA risk averse with index of risk aversion $\alpha > 0$.*

(P 4.1): *The unique symmetric equilibrium for the chore auction in increasing and differentiable strategies is given, for $t = 1, \ldots, N-K$, by*

$$\delta_t^\alpha(x; \mathbf{p}_{t-1}) = \frac{N-t}{(N-t+1)\,\alpha}\ln\left(S_t^\alpha(x)\right) - \frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i,$$

where

$$S^{\alpha}_{N-K}(x) = E\left[e^{\alpha Y^{(N)}_{N-K}} | Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}\right]$$

and, for $t < N-K$, $S^{\alpha}_{t}(x)$ *is defined recursively as*

$$S^{\alpha}_{t}(x) = E\left[\left(S^{\alpha}_{t+1}(Y^{(N)}_{t})\right)^{\frac{N-t-1}{N-t}} | Y^{(N)}_{t-1} > x > Y^{(N)}_{t}\right].$$

(P 4.2): *The unique symmetric equilibrium for the goods auction in increasing and differentiable strategies is given, for* $t = 1, \ldots, N-K$, *by*

$$\beta^{\alpha}_{t}(x; \mathbf{p}_{t-1}) = -\frac{N-t}{(N-t+1)\,\alpha} \ln\left(D^{\alpha}_{t}(x)\right) - \frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i$$

where

$$D^{\alpha}_{N-K}(x) = E\left[e^{-\alpha Z^{(N)}_{N-K}} | Z^{(N)}_{N-K} > x > Z^{(N)}_{N-K-1}\right]$$

and, for $t < N-K$, $D^{\alpha}_{k}(x)$ *is defined recursively as*

$$D^{\alpha}_{t}(x) = E\left[\left(D^{\alpha}_{t+1}(Z^{(N)}_{t})\right)^{\frac{N-t-1}{N-t}} | Z^{(N)}_{t} > x > Z^{(N)}_{t-1}\right].$$

In the remainder of this section we establish bounds on the CARA equilibrium bid functions and we compute the limiting equilibrium bid functions as bidders become infinitely risk averse.

Bounds and Comparative Statics

Proposition 5 establishes bounds for the CARA bid function.

Proposition 5: *For each* $\alpha > 0$ *and* $t = 1, \ldots, N-K$, *the bid functions* $\delta^{\alpha}_{t}(x; \mathbf{p}_{t-1})$ *and* $\beta^{\alpha}_{t}(x; \mathbf{p}_{t-1})$ *satisfy*

$$\delta^{0}_{t}(x; \mathbf{p}_{t-1}) < \delta^{\alpha}_{t}(x; \mathbf{p}_{t-1}) < \gamma_t(x; \mathbf{p}_{t-1}) < \beta^{\alpha}_{t}(x; \mathbf{p}_{t-1}) < \beta^{0}_{t}(x; \mathbf{p}_{t-1}) \ \forall x \in (0, \bar{x})$$

where

$$\gamma_t(x; \mathbf{p}_{t-1}) \equiv \frac{K}{N-t+1}(x - \frac{1}{K}\sum_{i=1}^{t-1} p_i).$$

The function γ_t provides an upper bound for bids in the chore auction and a lower bound for bids in the goods auction, and it has a natural interpretation. Consider the goods auction. The total surplus (as viewed by a bidder with value x) at round t available to the bidders who remain in the auction is $Kx - \sum_{i=1}^{t-1} p_i$. The bound γ_t is an equal share of this surplus divided among the $N - t + 1$ remaining bidders. In the goods auction, since $\beta_t^{\alpha}(x; \mathbf{p}_{t-1}) > \gamma_t(x; \mathbf{p}_{t-1})$, a bidder demands compensation of at least this amount. A similar interpretation applies for the chore auction.

Figure 2 below illustrates Proposition 5 when values are distributed $U[0, 1]$ and $K = 3$ and $N = 4$. The equilibrium bid functions for the chore auction are in red (solid line for $\alpha = 0$ and the dashed line for $\alpha = 10$). The analogous bid functions for the goods auction are shown in green. The bound $\gamma_1(x) = 3x/4$ is in black.

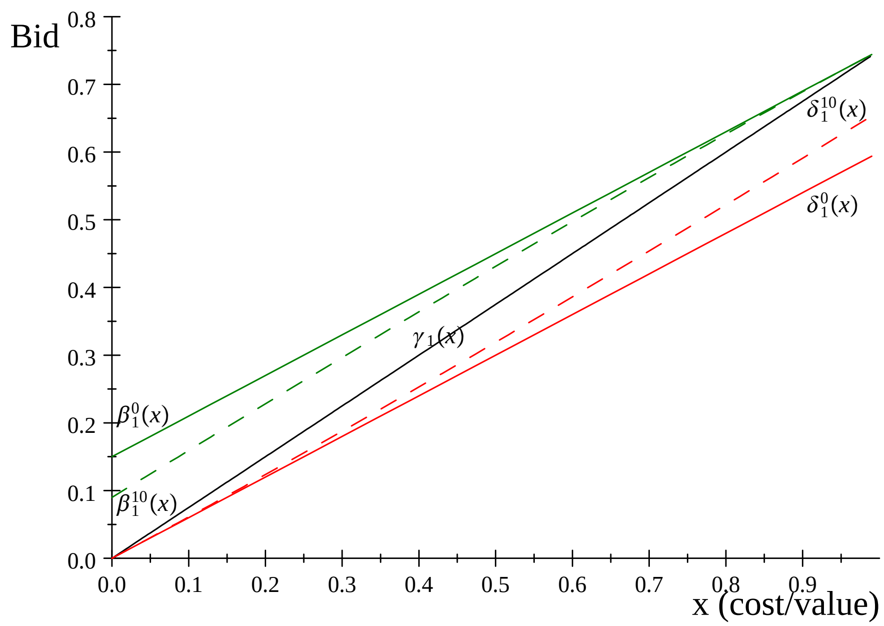

Fig. 2: Equilibrium bids and Risk Aversion

Proposition 6 establishes that in the chore auction bidders drop out earlier and pay more compensation to be excused from the chore as they are more risk averse. In the goods auction, bidders also drop out earlier but receive less compensation for surrendering their claim to a unit of the good as they are more risk averse. Although the problems of allocating K chores or K goods are not equivalent, remarkably the chore and the goods auctions have the same equilibrium bid function in the limit as bidders become infinitely risk averse. (Recall that the problem of allocating K goods can be equivalently framed as a problem of allocating $N - K$ chores.)

Proposition 6: *For each t, the bid function $\delta_t^\alpha(x; \mathbf{p}_{t-1})$ is increasing in α, the bid function $\beta_t^\alpha(x; \mathbf{p}_{t-1})$ is decreasing in α, and both have $\gamma_t(x; \mathbf{p}_{t-1})$ as their limit as α approaches infinity, i.e.,*

$$\lim_{\alpha\to\infty} \delta_t^\alpha(x; \mathbf{p}_{t-1}) = \lim_{\alpha\to\infty} \beta_t^\alpha(x; \mathbf{p}_{t-1}) = \gamma_t(x; \mathbf{p}_{t-1}) \ \forall x.$$

The function γ_t will play an important role in the next section.

4 Maxmin and Maxmin Perfect Strategies

In this section we take a decision-theoretic approach, and ask how bidders should behave in order to guarantee that they don't do "too badly." More precisely, we ask what strategy should a bidder follow in order to maximize his minimum payoff. For an active bidder at round t, let $v_t(\mathbf{x}, \boldsymbol{\delta}, \mathbf{p}_{t-1})$ be the bidder's payoff in the chore auction when $\mathbf{x} = (x_i, x_{-i})$ is the profile of values, $\boldsymbol{\delta} = (\delta^i, \delta^{-i})$ is the profile of strategies, and $\mathbf{p}_{t-1}$ is the sequence of dropout prices.

Definition: A strategy δ^i *guarantees* bidder i with value x_i a payoff of $\bar{v}_t$ at round t, given $\mathbf{p}_{t-1}$, if $v_t((x_i, x_{-i}), (\delta^i, \delta^{-i}), \mathbf{p}_{t-1}) \geq \bar{v}_t \ \forall x_{-i}, \delta^{-i}$.

Let $\bar{v}_t(x_i, \mathbf{p}_{t-1})$ be the largest payoff that bidder i with value x_i can guarantee in round t given $\mathbf{p}_{t-1}$.[3] Then $\bar{v}_1(x_i, \mathbf{p}_0)$ is the largest payoff that bidder i with value x_i can guarentee at the start of the auction.

Definition: A strategy $\bar{\delta}^i$ is a *maxmin strategy for bidder* i if $\bar{\delta}^i$ guarantees $\bar{v}_1(x_i, \mathbf{p}_0)$ for each $x_i \in [0, \bar{x}]$.

Proposition 7, which follows, shows that $\bar{v}_1(x_i, \mathbf{p}_0) = -Kx_i/N$ for the chore auction. It's easy to see that the strategy which calls for a bidder to drop out whenever the bid reaches Kx_i/N is a maxmin strategy: If a bidder drops, then he pays $-Kx_i/N$. If he never drops, it is because $N-K$ rivals dropped at bids (and pay compensation) greater than Kx_i/N. The total compensation is thus at least $(N-K)Kx_i/N$, of which bidder i receives $1/K$-th. Hence he receives compensation of at least $(N-K)x_i/N$ and his payoff is at least $(N-K)x_i/N - x_i = -Kx_i/N$. By the same reasoning, the same strategy is a maxmin strategy for the goods auction.

This strategy is simple in the sense that it does not depend on the prices at which rivals dropped at prior rounds. However, as the auction progresses, a bidder may be able to guarantee himself more than $-Kx_i/N$, e.g., if at round 1 a rival bidder drops at a bid above Kx_i/N. A maxmin perfect strategy maximizes a bidder's minimum payoff at every point along every path of play.

Definition: A strategy $\bar{\delta}^i$ is a *maxmin perfect strategy for bidder* i if $\bar{\delta}^i$ guarantees $\bar{v}_t(x_i, \mathbf{p}_{t-1})$ for each t, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{t-1}$.

Proposition 7 identifies the unique maxmin perfect strategy for the chore auction and shows that the same strategy is also the unique maxmin perfect strategy for the goods auction. The maxmin perfect strategy is, furthermore,

[3] Proposition 7 will establish that $\bar{v}_t(x_i, \mathbf{p}_{t-1})$ is well defined.

the strategy identified in Proposition 6 as the limit of the equilibrium bid functions as bidders become infinitely risk averse.

Proposition 7: *In both the chore auction and the goods auction, the strategy* γ^i*, given by*

$$\gamma_t^i(x_i; \mathbf{p}_{t-1}) = (Kx_i - \sum_{i=1}^{t-1} p_i)/(N - t + 1)$$

for each $t \in \{1, \ldots, N-1\}$*, and every* $x_i \in [0, \bar{x}]$ *and* $\mathbf{p}_{t-1}$ *is the unique maxmin perfect strategy. In the chore auction*

$$\bar{v}_t(x_i, \mathbf{p}_{t-1}) = -\gamma_t^i(x_i; \mathbf{p}_{t-1}).$$

In the goods auction the largest payoff that bidder i *with value* x_i *can guarantee in round* t *given* $\mathbf{p}_{t-1}$ *is* $-\bar{v}_t(x_i, \mathbf{p}_{t-1})$*.*

The intuition for this result is clear. Consider round $N - K$ in the chore auction following drop out prices $\mathbf{p}_{N-K-1}$. A bidder with value x_i whose strategy calls for him to drop at price p either drops at price p (and obtains $-p$) or a rival bidder drops at a higher price in which case he obtains a payoff of at least

$$\frac{1}{K}p + \frac{1}{K}\sum\nolimits_{i=1}^{N-K-1} p_i - x_i.$$

The bidder maximizes his minimum payoff by choosing p to equate these two payoffs. Solving for p yields $\gamma_{N-K}^i(x_i; \mathbf{p}_{t-1})$.

In the goods auction, by contrast, a bidder whose strategy calls for him to drop at price p either drops at price p (and obtains p) or a rival bidder drops at a lower price in which case he wins the item and he obtains a payoff of at least

$$-\left(\frac{1}{K}\sum\nolimits_{i=1}^{N-K-1} p_i + \frac{1}{K}p - x_i\right).$$

Again, the bidder maximizes his minimum payoff by choosing p to equate these two payoffs. It is immediate that the same p maximizes the bidder's minimum payoff in both the chore auction and the goods auction in the last round. An induction argument establishes the proposition.

5 Discussion

The present paper proposes two dynamic auctions for efficiently allocating chores and goods when participants have common obligations or claims. There are other ways of solving these kinds of problems. For example, the K units of the good could be sold to a third party for a price of $KE[Z_{N-K}^{(N)}]$, the proceeds to be shared equally among the N participants with a common claim, with the third party then selling the goods back to the participants via a Vickrey auction without a reserve. So long as bidders are risk neutral, in the dominant strategy equilibrium of the Vickrey auction, by our payoff equivalence result (Proposition 3) each bidder obtains the same payoff under this mechanism as in the goods and chore auctions. This solution, however, requires a third party to take on risk for a zero expected profit. The auctions we propose are simple, budget balanced, efficient and don't require a benevolent third party.

6 Appendix

This appendix contains the statement of Proposition 1′, which provides necessary and sufficient conditions for the goods auction and is the analog to Proposition 1 in the body of the paper for the chore auction. It also contains the proofs for our results on the chore auction. The proofs for the goods auction are symmetric to the proofs for the chore auction, and are relegated to the Supplemental Appendix.

Proposition 1′: *(i) Any symmetric equilibrium* β *of the goods auction in* *increasing and differentiable bid strategies, satisfies the following system of differential equations:*

$$u'(\beta_{N-K}(x|\mathbf{p}_{N-K-1}))\beta'_{N-K}(x|\mathbf{p}_{N-K-1})$$
$$= [u(\beta_{N-K}(x|\mathbf{p}_{N-K-1})) - u(x - \frac{1}{K}(\beta_{N-K}(x|\mathbf{p}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i))]\lambda^N_{N-K}(x),$$

and, for $t \in \{1, \ldots, N-K-1\}$*, that*

$$\begin{aligned} & u'\left(\beta_t(x;\mathbf{p}_{t-1})\right)\beta'_t(x;\mathbf{p}_{t-1}) \\ = \quad & [u\left(\beta_t(x;\mathbf{p}_{t-1})\right) - u(\beta_{t+1}(x;\mathbf{p}_{t-1},\beta_t(x;\mathbf{p}_{t-1})))]\lambda^N_t(x), \end{aligned}$$

where

$$\lambda^N_t(x) = (N-t)\frac{f(x)}{1-F(x)}.$$

(ii) If $\beta = (\beta_1, \ldots, \beta_{N-K})$ *is a solution to the system of differential equations in* (i)*, then it is an equilibrium.*

The proof of Proposition 1 follows.

Proof of Proposition 1: Let $\delta = (\delta_1, \ldots, \delta_{N-K})$ be a symmetric equilibrium in increasing and differentiable strategies. For each $t \leq N-K$, let $\pi_t(\hat{x}, x|\mathbf{y}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is $\hat{x}$ (i.e., he bids $\delta_t(\hat{x}|\mathbf{y}_{t-1})$), when $\mathbf{y}_{t-1}$ is the profile of values of the $t-1$ bidders to drop in prior rounds. In this case we will sometimes say the bidder "bids $\hat{x}$". Let

$$\Pi_t(x|\mathbf{y}_{t-1}) = \pi_t(x, x|\mathbf{y}_{t-1})$$

be the bidder's equilibrium payoff in round t:

(a) For each $\mathbf{y}_{t-1}$:

(a.i) δ_t satisfies the differential equation given in Proposition 1(i).

(a.ii) if $x \leq \mathbf{y}_{t-1}$ then $x \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{y}_{t-1})$, i.e., it is optimal for each bidder to follow δ_t in round t; if $x > \mathbf{y}_{t-1}$ then $y_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{y}_{t-1})$.

(b) For each $\mathbf{y}_{t-1}$:

$$\frac{d\Pi_t(x|\mathbf{y}_{t-1})}{dx} \leq 0\,.$$

We prove by induction that the claim is true for each $t \in \{1, \ldots, N-K\}$, thereby establishing Proposition 1. Note that since equilibrium is in increasing strategies, at any round t the sequence of dropout prices $(p_1, \ldots, p_{t-1})$ reveals the $t-1$ highest values $\mathbf{y}_{t-1} = (y_1, \ldots, y_{t-1})$.

We first show the claim is true for round $N-K$. Let $\mathbf{y}_{N-K-1}$ be arbitrary and consider an active bidder whose value is x but who bids as though it is $\hat{x} \leq y_{N-K-1}$. There are two cases to consider: (i) $x \leq y_{N-K-1}$ and (ii) $x > y_{N-K-1}$.

Case (i): $x \leq y_{N-K-1}$. With a bid of $\hat{x} \leq y_{N-K-1}$, if $\hat{x} < y_{N-K}$ then a rival bidder drops out first at the price $\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1})$, the bidder undertakes the chore, and he receives compensation of

$$\frac{1}{K}\left(\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right).$$

If $\hat{x} > y_{N-K}$ then the bidder drops before any rival and he pays compensation $\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})$. Hence $\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1}) =$

$$\int_0^{\hat{x}} u(-\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}))g_{N-K}^{(N-1)}(y_{N-K}|y_{N-K-1})dy_{N-K}$$
$$+ \int_{\hat{x}}^{y_{N-K-1}} u\left(\tfrac{1}{K}\left(\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i\right) - x\right) g_{N-K}^{(N-1)}(y_{N-K}|y_{N-K-1})dy_{N-K}.$$

Differentiating with respect to $\hat{x}$ yields $\partial\pi_{N-K}(\hat{x},x|\mathbf{y}_{N-K-1})/\partial\hat{x} =$

$$\begin{aligned}&-u'(-\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}))\delta'_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})G^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1})\\&+u(-\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}))g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1})\\&-u\left(\tfrac{1}{K}\left(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})+\textstyle\sum_{i=1}^{N-K-1}p_i\right)-x\right)g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1}).\end{aligned} \tag{1}$$

A necessary condition for δ to be an equilibrium is that $\partial\pi_{N-K}(\hat{x},x|\mathbf{z}_{K+1})/\partial\hat{x}|_{\hat{x}=x} = 0$, i.e.,

$$\begin{aligned}&u'(-\delta_{N-K}(x|\mathbf{y}_{N-K-1}))\delta'_{N-K}(x|\mathbf{y}_{N-K-1})\\&=[u(-\delta_{N-K}(x|\mathbf{y}_{N-K-1}))-u\left(\tfrac{1}{K}(\delta_{N-K}(x|\mathbf{y}_{N-K-1})+\sum_{i=1}^{N-K-1}p_i)-x\right)]\bar{\lambda}^N_{N-K}(x).\end{aligned}$$

where

$$\frac{g^{(N-1)}_{N-K}(x|y_{N-K-1})}{G^{(N-1)}_{N-K}(x|y_{N-K-1})} = K\frac{f(x)}{F(x)} = \bar{\lambda}^N_{N-K}(x).$$

Alternatively, since types can be inferred from dropout prices, we can write the necessary condition as

$$\begin{aligned}&u'(-\delta_{N-K}(x|\mathbf{p}_{N-K-1}))\delta'_{N-K}(x|\mathbf{p}_{N-K-1})\\&=[u(-\delta_{N-K}(x|\mathbf{p}_{N-K-1}))-u\left(\tfrac{1}{K}(\delta_{N-K}(x|\mathbf{p}_{N-K-1})+\sum_{i=1}^{N-K-1}p_i)-x\right)]\bar{\lambda}^N_{N-K}(x),\end{aligned}$$

which establishes (a.i) for $t = N-K$.

The necessary condition holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$\begin{aligned}&u'(-\delta_{N-K}(\hat{x}|y_{N-K-1}))\delta'_{N-K}(\hat{x}|y_{N-K-1})\\&=[u(-\delta_{N-K}(\hat{x}|y_{N-K-1}))-u(\tfrac{1}{K}(\delta_{N-K}(\hat{x}|y_{N-K-1})+\textstyle\sum_{i=1}^{N-K-1}p_i)-\hat{x})]\bar{\lambda}^N_{N-K}(\hat{x}).\end{aligned} \tag{2}$$

Substituting (2) into (1) and simplifying yields

$$\frac{\partial\pi_{N-K}(\hat{x},x|\mathbf{y}_{N-K-1})}{\partial\hat{x}} = \left[\begin{array}{c}u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})+\sum_{i=1}^{N-K-1}p_i)-\hat{x})\\-u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1})+\sum_{i=1}^{N-K-1}p_i)-x)\end{array}\right]g^{(N-1)}_{N-K}(\hat{x}|y_{N-K-1}).$$

Clearly, $\partial\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})/\partial y|_{\hat{x}=x} = 0$. Moreover, for $\hat{x} \leq y_{N-K-1}$ we have

$$\frac{\partial^2\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial\hat{x}\partial x} = u'\left(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x\right) g_{N-K}^{(N-1)}(\hat{x}|y_{N-K-1}) \geq 0,$$

where the inequality holds since $u' > 0$ and $g_{N-K}^{(N-1)}(\hat{x}|y_{N-K-1}) \geq 0$. Hence, if $x \leq y_{N-K-1}$ then $x \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})$ by Lemma 0 of McAfee (1992).

Case (ii): $x > y_{N-K-1}$. It is clearly never optimal for a bidder to bid as though his type is greater than y_{N-K-1}, i.e., bid more than $\delta_{N-K}(y_{N-K-1}|\mathbf{y}_{N-K-1})$, since he pays less compensation with a bid of $\delta_{N-K}(y_{N-K-1}|\mathbf{y}_{N-K-1})$. (For either bid, he drops out for sure since the other remaining bidders all have costs below y_{N-K-1}.)

For $\hat{x} \leq y_{N-K-1}$ we have

$$\frac{\partial\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial\hat{x}} = \begin{bmatrix} u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - \hat{x}) \\ -u(\frac{1}{K}(\delta_{N-K}(\hat{x}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x) \end{bmatrix} g_{N-K}^{(N-1)}(\hat{x}|y_{N-K-1}) > 0$$

and thus $y_{N-K-1} \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})$. Hence (a.ii) is true for $t = N - K$.

To prove (b), note that $d\Pi_{N-K}(x|\mathbf{y}_{N-K-1})/dx$ is

$$\begin{aligned} & \left.\frac{\partial\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial\hat{x}}\right|_{\hat{x}=x} + \left.\frac{\partial\pi_{N-K}(\hat{x}, x|\mathbf{y}_{N-K-1})}{\partial x}\right|_{\hat{x}=x} \\ = & -\int_{\hat{x}}^{y_{N-K-1}} u'\left(\frac{1}{K}(\delta_{N-K}(y_{N-K}|\mathbf{y}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i) - x\right) g_{N-K}^{(N-1)}(y_{N-K}|y_{N-K-1}) dy_{N-K} \\ \leq & \ 0, \end{aligned}$$

where the second equality holds since $\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial y|_{\hat{x}=x} = 0$. Hence (b) holds for $t = N - K$.

Assume the claim is true for rounds $t+1$ through $N-K$. We show it is true for round t. Let $\mathbf{y}_{t-1}$ be arbitrary. If $x > y_{t-1}$ then, by the same argument as before, $y_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{y}_{t-1})$.

Suppose $x \leq y_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $\hat{x} \leq y_{t-1}$. We need to distinguish between two cases: (i) $\hat{x} \in [x, y_{t-1}]$ and (ii) $\hat{x} < x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of $\hat{x}$ as $\pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})$ if $\hat{x} \in [x, y_{t-1}]$ and as $\pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})$ if $\hat{x} < x$.

Case (i): Suppose $\hat{x} \in [x, y_{t-1}]$. If $y_t \in [\hat{x}, y_{t-1}]$ the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t)$. If $y_t \leq \hat{x}$ he pays compensation of $\delta_t(\hat{x}|\mathbf{y}_{t-1})$. Hence his payoff is

$$\begin{aligned}\pi_t^H(\hat{x}, x|\mathbf{y}_{t-1}) = \;& \textstyle\int_{\hat{x}}^{y_{t-1}} \Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t) g_t^{(N-1)}(y_t|y_{t-1}) dy_t \\ & + \textstyle\int_0^{\hat{x}} u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) dy_t.\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields $\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x} =$

$$\begin{aligned}&-\Pi_{t+1}(x|\mathbf{y}_{t-1}, \hat{x}) g_t^{(N-1)}(\hat{x}|y_{t-1}) + u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(\hat{x}|y_{t-1}) \\ &-u'\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) \delta_t'(\hat{x}|\mathbf{y}_{t-1}) G_t^{(N-1)}(\hat{x}|y_{t-1}).\end{aligned}$$

Since

$$\Pi_{t+1}(x|\mathbf{y}_{t-1}, x) = u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x)),$$

and

$$\frac{g_t^{(N-1)}(x|y_{t-1})}{G_t^{(N-1)}(x|y_{t-1})} = (N-t)\frac{f(x)}{F(x)} = \bar{\lambda}_t^N(x),$$

the necessary condition for equilibrium that $\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \leq 0$ can be written as

$$\left[u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))\right] \bar{\lambda}_t^N(x) \leq u'\left(-\delta_t(x|\mathbf{y}_{t-1})\right) \delta_t'(x|\mathbf{y}_{t-1}). \tag{3}$$

Also, for $\hat{x} \in [x, y_{t-1}]$ we have

$$\frac{\partial^2 \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})}{\partial x \partial \hat{x}} = -\frac{d\Pi_{t+1}(x|\mathbf{y}_{t-1}, \hat{x})}{dx} g_t^{(N-1)}(\hat{x}|y_{t-1}) \geq 0,$$

where the inequality follows since (b) is true for round $t+1$ by the induction hypothesis.

Case (ii): Suppose $\hat{x} < x$. If $y_t \in [x, y_{t-1}]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t)$. If $y_t \in [\hat{x}, x]$, then he continues to round $t+1$ and, by the induction hypothesis, he bids y_t and pays compensation of $\delta_{t+1}(y_t|\mathbf{y}_{t-1}, y_t)$. If $y_t < \hat{x}$ then in round t he pays compensation of $\delta_t(\hat{x}|\mathbf{y}_{t-1})$. His payoff at round t is therefore

$$\begin{aligned}\pi_t^L(\hat{x}, x|\mathbf{y}_{t-1}) = \quad & \textstyle\int_x^{y_{t-1}} \Pi_{t+1}(x|\mathbf{y}_{t-1}, y_t) g_t^{(N-1)}(y_t|y_{t-1}) dy_t \\ & + \textstyle\int_{\hat{x}}^{x} u(-\delta_{t+1}(y_t|\mathbf{y}_{t-1}, y_t)) g_t^{(N-1)}(y_t|y_{t-1}) dy_t, \\ & + \textstyle\int_0^{\hat{x}} u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) dy_t.\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields

$$\begin{aligned}\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x} \quad = \quad & -u(-\delta_{t+1}(\hat{x}|\mathbf{y}_{t-1}, \hat{x})) g_t^{(N-1)}(\hat{x}|y_{t-1}) + u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1}) \\ & -u'\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) \delta_t'(\hat{x}|\mathbf{y}_{t-1}) G_t^{(N-1)}(\hat{x}|y_{t-1}).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t^L(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \geq 0$, i.e.,

$$u'\left(-\delta_t(x|\mathbf{y}_{t-1})\right) \delta_t'(x|\mathbf{y}_{t-1}) \leq [u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))] \bar{\lambda}_t^N(x). \tag{4}$$

Equations (3) and (4) imply that

$$u'\left(-\delta_t(x|\mathbf{y}_{t-1})\right) \delta_t'(x|\mathbf{y}_{t-1}) = [u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(x|\mathbf{y}_{t-1}, x))] \bar{\lambda}_t^N(x). \tag{5}$$

Hence (3) holds as an equality, i.e., $\partial \pi_t^H(\hat{x}, x|\mathbf{y}_{t-1})/\partial \hat{x}|_{\hat{x}=x} = 0$.

Since the bid functions are increasing, we can replace $\mathbf{y}_{t-1}$ with $\mathbf{p}_{t-1}$ and replace $\delta_{t+1}(x|\mathbf{y}_{t-1}, x)$ with $\delta_{t+1}(x; \mathbf{p}_{t-1}, \delta_t(x; \mathbf{p}_{t-1}))$, writing the first order condition as

$$
\begin{aligned}
&u'\left(-\delta_t(x;\mathbf{p}_{t-1})\right)\delta_t'(x;\mathbf{p}_{t-1}) \\
&= [u\left(-\delta_t(x;\mathbf{p}_{t-1})\right) - u(-\delta_{t+1}(x;\mathbf{p}_{t-1},\delta_t(x;\mathbf{p}_{t-1})))]\bar{\lambda}_t^N(x),
\end{aligned}
$$

which establishes (a.i) for round t.

Equation (5) holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$
u'\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right)\delta_t'(\hat{x}|\mathbf{y}_{t-1}) = [u\left(-\delta_t(\hat{x}|\mathbf{y}_{t-1})\right) - u(-\delta_{t+1}(\hat{x}|\mathbf{y}_{t-1},\hat{x}\,))]\bar{\lambda}_t^N(\hat{x}).
$$

Substituting this expression into the expression for $\partial\pi_t^L(\hat{x},x|\mathbf{y}_{t-1})/\partial\hat{x}$ yields

$$
\frac{\partial\pi_t^L(\hat{x},x|\mathbf{y}_{t-1})}{\partial\hat{x}} = 0 \text{ for } \hat{x} \leq x.
$$

Furthermore,

$$
\frac{\partial^2\pi_t^L(\hat{x},x|\mathbf{y}_{t-1})}{\partial\hat{x}\partial x} = 0 \text{ for } \hat{x} \geq x.
$$

We have shown that

$$
\left.\frac{\partial\pi_t^L(\hat{x},x|\mathbf{y}_{t-1})}{\partial\hat{x}}\right|_{\hat{x}=x} = \left.\frac{\partial\pi_t^H(\hat{x},x|\mathbf{y}_{t-1})}{\partial\hat{x}}\right|_{\hat{x}=x} = 0
$$

and

$$
\frac{\partial^2\pi_t^H(\hat{x},x|\mathbf{y}_{t-1})}{\partial\hat{x}\partial x} \geq 0 \text{ for } \hat{x} \in [x, y_{t-1}] \text{ and } \frac{\partial^2\pi_t^L(\hat{x},x|\mathbf{y}_{t-1})}{\partial\hat{x}\partial x} \geq 0 \text{ for } \hat{x} < x.
$$

Hence (a.ii) is true for round t by Van Essen and Wooders' (2016) extension of McAfee's (1992) Lemma 0.

To establish (b) is true for round t, observe that

$$
\Pi_t(x|\mathbf{y}_{t-1}) = \begin{array}{l} \int_x^{y_{t-1}} \Pi_{t+1}(x|\mathbf{y}_{t-1},y_t) g_t^{(N-1)}(y_t|y_{t-1})dy_t \\ + \int_0^x u\left(-\delta_t(x|\mathbf{y}_{t-1})\right) g_t^{(N-1)}(y_t|y_{t-1})dy_t. \end{array}
$$

Differentiating and simplifying yields

$$
\frac{d\Pi_t(x|\mathbf{y}_{t-1})}{dx} = \int_x^{y_{t-1}} \frac{d\Pi_{t+1}(x|\mathbf{y}_{t-1},y_t)}{dx} g_t^{(N-1)}(y_t|y_{t-1})dy_t \leq 0,
$$

where the equality follows from $\Pi_{t+1}(x|\mathbf{y}_{t-1},x) = u(-\delta_{t+1}(x|\mathbf{y}_{t-1},x))$ and (5), and the inequality follows since $d\Pi_{t+1}(x|\mathbf{y}_{t-1},y_t)/dx \leq 0$ by the induction hypothesis. □

Proof of Proposition 2.1: The proof is by induction. By Proposition 1(i), at round $N-K$ the differential equation for the equilibrium bid function is

$$\delta'_{N-K}(x;\mathbf{p}_{N-K-1}) = -\left[\frac{K+1}{K}\delta_{N-K}(x;\mathbf{p}_{N-K-1}) + \frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right]\bar{\lambda}^N_{N-K}(x).$$

Multiplying both sides by $F(x)$ we obtain

$$\delta'_{N-K}(x;\mathbf{p}_{N-K-1})F(x)+(K+1)\delta_{N-K}(x;\mathbf{p}_{N-K-1})f(x) = \left[x - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right]Kf(x),$$

i.e.,

$$\frac{d}{dx}\left(\delta_{N-K}(x;\mathbf{p}_{N-K-1})F(x)^{K+1}\right) = \left[x - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right]Kf(x)F(x)^K.$$

By the Fundamental Theorem of Calculus we have

$$\delta_{N-K}(x;\mathbf{p}_{N-K-1})F(x)^{K+1} = \int_0^x \left[s - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right]Kf(s)F(s)^K ds + C.$$

The LHS of this equation is zero when $x=0$ (since $F(0)=0$), which implies $C=0$. Hence

$$\delta_{N-K}(x;\mathbf{p}_{N-K-1}) = \int_0^x \left[s - \frac{1}{K}\sum_{i=1}^{N-K-1} p_i\right]K\frac{F(s)^K}{F(x)^{K+1}}f(s)ds.$$

Since

$$\int_0^x s(K+1)\frac{F(s)^K}{F(x)^{K+1}}f(s)ds = E[Y^{(N)}_{N-K}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}],$$

then

$$\delta_{N-K}(x;\mathbf{p}_{N-K-1}) = \frac{K}{K+1}E[Y_{N-K}^{(N)}|Y_{N-K-1}^{(N)} > x > Y_{N-K}^{(N)}] - \frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i,$$

which establishes the result for round $N-K$.

Assume in round t that

$$\delta_t(x;\mathbf{p}_{t-1}) = \frac{K}{N-t+1}E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] - \frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i.$$

We need to show that $\delta_{t-1}(x;\mathbf{p}_{t-2})$ is as given in Proposition 2(i). The differential equation for round $t-1$ is

$$\delta'_{t-1}(x;\mathbf{p}_{t-2}) = [-\delta_{t-1}(x;\mathbf{p}_{t-2}) + \delta_t(x;\mathbf{p}_{t-2},\delta_{t-1}(x;\mathbf{p}_{t-2}))]\bar{\lambda}_{t-1}(x).$$

By the induction hypothesis

$$\begin{aligned}\delta_t(x;\mathbf{p}_{t-2},\delta_{t-1}(x;\mathbf{p}_{t-2})) &= \frac{K}{N-t+1}E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] \\ &\quad -\frac{1}{N-t+1}(\sum_{i=1}^{t-2} p_i + \delta_{t-1}(x;\mathbf{p}_{t-2})).\end{aligned}$$

Hence

$$\delta'_{t-1}(x;\mathbf{p}_{t-2}) = \left[\begin{array}{c} -\frac{N-t+2}{N-t+1}\delta_{t-1}(x;\mathbf{p}_{t-2}) + \frac{K}{N-t+1}E\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] \\ -\frac{1}{N-t+1}\sum_{i=1}^{t-2} p_i \end{array}\right]\bar{\lambda}_{t-1}(x).$$

Multiplying both sides by $F(x)^{N-t+2}$ yields

$$\frac{d}{dx}\left(\delta_{t-1}(x;\mathbf{p}_{t-2})F(x)^{N-t+2}\right) = \left[KE\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > x > Y_t^{(N)}\right] - \sum_{i=1}^{t-2} p_i\right]F(x)^{N-t+1}f(x).$$

By the Fundamental Theorem of Calculus and since $F(0)=0$ then

$$\delta_{t-1}(x;\mathbf{p}_{t-2})F(x)^{N-t+2} = \int_0^x \left[KE\left[Y_{N-K}^{(N)}|Y_{t-1}^{(N)} > s > Y_t^{(N)}\right] - \sum_{i=1}^{t-2} p_i\right]F(s)^{N-t+1}f(s)ds.$$

Hence

$$\delta_{t-1}(x;\mathbf{p}_{t-2}) = \int_0^x \left[KE\left[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > s > Y_t^{(N)}\right] - \sum_{i=1}^{t-2} p_i \right] f(s) \frac{F(s)^{N-t+1}}{F(x)^{N-t+2}} ds.$$

Since (to be established momentarily)

$$\begin{aligned} & \int_0^x E\left[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > s > Y_t^{(N)}\right] (N-t+2) f(s) \frac{F(s)^{N-t+1}}{F(x)^{N-t+2}} ds \\ = \quad & E\left[Y_{N-K}^{(N)} | Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right] \end{aligned}$$

then

$$\delta_{t-1}(x;\mathbf{p}_{t-2}) = \frac{K}{N-t+2} \left(E\left[Y_{N-K}^{(N)} | Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right] - \frac{1}{K} \sum_{i=1}^{t-2} p_i \right),$$

which completes the proof.

Finally, we establish the equality just used. We have

$$\begin{aligned} & \int_0^x E\left[Y_{N-K}^{(N)} | Y_{t-1}^{(N)} > s > Y_t^{(N)}\right] (N-t+2) f(s) \frac{F(s)^{N-t+1}}{F(x)^{N-t+2}} ds \\ = & \int_0^x \left(\int_0^s r \frac{(N-t+1)!}{K!(N-t-K)!} \frac{F(r)^K [F(s)-F(r)]^{N-t-K}}{F(s)^{N-t+1}} f(r) dr \right) (N-t+2) f(s) \frac{F(s)^{N-t+1}}{F(x)^{N-t+2}} ds \\ = & \int_0^x r \frac{(N-t+1)!}{K!(N-t-K)!} \int_r^x \frac{F(r)^K [F(s)-F(r)]^{N-t-K}}{F(x)^{N-t+2}} (N-t+2) f(r) f(s) ds dr \\ = & \int_0^x r \frac{(N-t+2)!}{K!(N-t-K+1)!} \frac{F(r)^K [F(x)-F(r)]^{N-t-K+1}}{F(x)^{N-t+2}} f(r) dr \\ = & E\left[Y_{N-K}^{(N)} | Y_{t-2}^{(N)} > x > Y_{t-1}^{(N)}\right]. \ \square \end{aligned}$$

Proof of Proposition 3: By the Revelation Principle, for any mechanism and BNE of the mechanism, there is an equivalent direct mechanism with truthtelling as a BNE. Hence it is without loss of generality to take ξ to be a direct mechanism.

Let $m_\xi(x)$ be the expected payment of a bidder who reports value x when all the other bidders report their values truthfully. Since the bidders with the

K highest reports win an item, the expected payoff of a bidder with value x who reports $\hat{x}$ is

$$\pi(\hat{x}, x) = H_{N-K}^{(N-1)}(\hat{x})x - m_\xi(\hat{x}),$$

where $H_{N-K}^{(N-1)}(\hat{x}) = \Pr(Z_{N-K}^{(N-1)} < \hat{x})$ is the probability that the $N-K$-th lowest of $N-1$ values is less than $\hat{x}$.

Differentiating $\pi(\hat{x}, x)$ with respect to $\hat{x}$ yields

$$\frac{d\pi(\hat{x}, x)}{d\hat{x}} = h_{N-K}^{(N-1)}(\hat{x})x - m'_\xi(\hat{x}).$$

Since truthtelling is a BNE then

$$m'_\xi(x) = h_{N-K}^{(N-1)}(x)x \text{ for all } x \in [0, \bar{x}].$$

By the Fundamental Theorem of Calculus we have

$$m_\xi(x) = m_\xi(0) + \int_0^x t h_{N-K}^{(N-1)}(t)dt.$$

Since the mechanism is budget balanced, the ex-ante expected payment of a bidder is zero, i.e.,

$$\int_0^{\bar{x}} m_\xi(x)f(x)dx = m_\xi(0) + \int_0^{\bar{x}} \left[\int_0^x t h_{N-K}^{(N-1)}(t)dt\right] f(x)dx = 0.$$

The double integral above is

$$\frac{1}{N}\int_0^{\bar{x}} \left[\int_0^x t\frac{N!}{(N-K-1)!\,(K-1)!}F(t)^{N-K-1}[1-F(t)]^{K-1}f(t)dt\right] f(x)dx.$$

Reversing the order of integration and factoring out K yields

$$\begin{aligned}
& \frac{K}{N}\int_0^{\bar{x}}\int_t^{\bar{x}} t\frac{N!}{(N-K-1)!K!}F(t)^{N-K-1}[1-F(t)]^{K-1}f(x)f(t)dxdt \\
= & \frac{K}{N}\int_0^{\bar{x}} t\frac{N!}{(N-K-1)!K!}F(t)^{N-K-1}[1-F(t)]^{K}f(t)dt \\
= & \frac{K}{N}E[Z_{N-k}^{(N)}].
\end{aligned}$$

Hence
$$m_\xi(0) = -\frac{K}{N}E[Z^{(N)}_{N-k}].$$
Thus the expected payoff to a bidder with value x is
$$H^{(N-1)}_{N-K}(x)x - m_\xi(x) = H^{(N-1)}_{N-K}(x)x - \left(-\frac{K}{N}E[Z^{(N)}_{N-k}] + \int_0^x th^{(N-1)}_{N-K}(t)dt\right).$$
Integrating the RHS by parts, we have
$$H^{(N-1)}_{N-K}(x)x - m_\xi(x) = \frac{K}{N}E[Z^{(N)}_{N-k}] + \int_0^x H^{(N-1)}_{N-K}(t)dt,$$
which establishes the result. □

Proof of Proposition 4.1: To save space we write δ_t rather than $\delta_t(x;\mathbf{p}_{t-1})$. At round $t = N-K$, the differential equation that characterizes equilibrium behavior is
$$\frac{d}{dx}\left(e^{\alpha\frac{K+1}{K}\delta_{N-K}}F(x)^{K+1}\right) = e^{-\alpha\left(\frac{1}{K}\sum_{i=1}^{N-K-1}p_i - x\right)}(K+1)F(x)^K f(x).$$
From the Fundamental Theorem of Calculus, we have
$$e^{\alpha\frac{K+1}{K}\delta_{N-K}}F(x)^{K+1} = \int_0^x e^{-\alpha\left(\frac{1}{K}\sum_{i=1}^{N-K-1}p_i - s\right)}(K+1)F(s)^K f(s)ds + C.$$
At $x = 0$, the LHS of the above equation is equal to zero and hence $C = 0$. So
$$e^{\alpha\frac{K+1}{K}\delta_{N-K}} = e^{-\alpha\left(\frac{1}{K}\sum_{i=1}^{N-K-1}p_i\right)}\frac{\int_0^x e^{\alpha z}(K+1)F(z)^K f(z)dz}{F(x)^{K+1}}.$$
Taking logs of both sides we have
$$\alpha\frac{K+1}{K}\delta_{N-K} = -\alpha\frac{1}{K}\left(\sum_{i=1}^{N-K-1}p_i\right) + \ln\left(\frac{\int_0^x e^{\alpha z}(K+1)F(z)^K f(z)dz}{F(x)^{K+1}}\right),$$

and hence

$$\begin{aligned}
\delta_{N-K}\left(x;\mathbf{p}_{N-K-1}\right) &= \frac{K}{(K+1)\,\alpha}\ln\left(\frac{\int_0^x e^{\alpha z}(K+1)F(z)^K f(z)dz}{F(x)^{K+1}}\right)-\frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i\\
&= \frac{K}{(K+1)\,\alpha}\ln\left(E\left[e^{\alpha Y_{N-K}^{(N)}}|Y_{N-K-1}^{(N)}>x>Y_{N-K}^{(N)}\right]\right)-\frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i\\
&= \frac{K}{(K+1)\,\alpha}\ln\left(S_{N-K}^{\alpha}(x)\right)-\frac{1}{K+1}\sum_{i=1}^{N-K-1}p_i.
\end{aligned}$$

Next, we solve for the round $t-1$ bid function. Assume that in round $t \leq N-K$, bidders follow the bid function

$$\delta_t\left(x;\mathbf{p}_{t-1}\right)=\frac{N-t}{(N-t+1)\,\alpha}\ln\left(S_t^{\alpha}(x)\right)-\frac{1}{N-t+1}\sum_{i=1}^{t-1}p_i.$$

Note that this implies that $\delta_t\left(x;\mathbf{p}_{t-2},\delta_{t-1}\left(x;\mathbf{p}_{t-2}\right)\right)=$

$$\frac{N-t}{(N-t+1)\,\alpha}\ln\left(S_t^{\alpha}(x)\right)-\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i-\frac{1}{N-t+1}\delta_{t-1}\left(x;\mathbf{p}_{t-2}\right).$$

After some manipulation, the differential equation for round $t-1$ from Proposition 1 can be written as

$$\frac{d}{dx}\left(e^{\alpha\frac{N-t+2}{N-t+1}\delta_{t-1}}F(x)^{N-t+2}\right)=e^{-\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}S_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}(N-t+2)F(x)^{N-t+1}f(x).$$

From the Fundamental Theorem of Calculus we have

$$e^{\alpha\frac{N-t+2}{N-t+1}\delta_{t-1}}F(x)^{N-t+2}=\int_0^x e^{-\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}S_t^{\alpha}(s)^{\frac{N-t}{N-t+1}}(N-t+2)F(s)^{N-t+1}f(s)ds+C.$$

At $x=0$, the LHS of the above equation is equal to zero and hence $C=0$. Rearranging yields $\delta_{t-1}\left(x;\mathbf{p}_{t-2}\right)=$

$$\begin{aligned}
&\frac{N-t+1}{(N-t+2)\,\alpha}\ln\left(\frac{\int_0^x S_t^{\alpha}(s)^{\frac{N-t}{N-t+1}}(N-t+2)F(s)^{N-t+1}f(s)ds}{F(x)^{N-t+2}}\right)-\frac{1}{N-t+2}\sum_{i=1}^{t-2}p_i\\
=\ &\frac{N-t+1}{(N-t+2)\,\alpha}\ln\left(S_{t-1}^{\alpha}(x)\right)-\frac{1}{N-t+2}\sum_{i=1}^{t-2}p_i,
\end{aligned}$$

where second equality holds since

$$S^{\alpha}_{t-1}(x) = E\left[S^{\alpha}_{t}(Y^{(N)}_{t-1})^{\frac{N-t}{N-t+1}}|Y^{(N)}_{t-2} > x > Y^{(N)}_{t-1}\right].\Box$$

Proof of Proposition 5: We establish the inequalities for the chore auction here, and leave the proofs for the goods auction to the Supplemental Appendix.

We show that for each $t = 1, \ldots, N-K$ and $\mathbf{p}_{t-1}$ that $\delta^0_t(x;\mathbf{p}_{t-1}) < \delta^{\alpha}_t(x;\mathbf{p}_{t-1})$ for $x > 0$. The proof is by induction. For $t = N-K$, since e^x is a convex function, then by Jensen's Inequality, for $x > 0$ we have

$$e^{E[\alpha Y^{(N)}_{N-K}|Y^{(N)}_{N-K}<x<Y^{(N)}_{N-K-1}]} < E[e^{\alpha Y^{(N)}_{N-K}}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}].$$

Noting that the RHS is $S^{\alpha}_{N-K}(x)$, taking the log of both sides and multiplying through by $K/((K+1)\,\alpha)$ yields

$$\frac{K}{K+1}E[Y^{(N)}_{N-K}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}] < \frac{K}{(K+1)\alpha}\ln(S^{\alpha}_{N-K}(x)).$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$ to both sides yields $\delta^0_{N-K}(x;\mathbf{p}_{N-K-1}) < \delta^{\alpha}_{N-K}(x;\mathbf{p}_{N-K-1})$ for $x > 0$.

For $t \le N-K$, define

$$\Delta^0_t(x) = \frac{K}{N-t+1}E[Y^{(N)}_{N-K}|Y^{(N)}_{t-1} > x > Y^{(N)}_t],$$

and

$$\Delta^{\alpha}_t(x) = \frac{1}{\alpha}\ln\left(S^{\alpha}_t(x)^{\frac{N-t}{N-t+1}}\right),$$

where $S^{\alpha}_t(x)$ is defined in P4.1. We have that

$$e^{-\alpha\Delta^{\alpha}_t(x)} = S^{\alpha}_t(x)^{\frac{N-t}{N-t+1}}.$$

We established above that $\Delta^0_{N-K}(x) < \Delta^{\alpha}_{N-K}(x)$.

Assume for $t \leq N-K-1$ that $\Delta^0_{t+1}(x) < \Delta^\alpha_{t+1}(x)$ for $x > 0$. We show that $\Delta^0_t(x) < \Delta^\alpha_t(x)$ for $x > 0$. Since $\alpha\Delta^0_{t+1}(x) < \alpha\Delta^\alpha_{t+1}(x)$ and e^x is increasing, then

$$e^{\alpha\Delta^0_{t+1}(x)} < e^{\alpha\Delta^\alpha_{t+1}(x)} \text{ for } x > 0,$$

or

$$e^{\alpha\Delta^0_{t+1}(x)} < S^\alpha_{t+1}(x)^{\frac{N-t-1}{N-t}} \text{ for } x > 0.$$

Thus

$$E[e^{\alpha\Delta^0_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t] < E[S^\alpha_{t+1}(Y^{(N)}_t)^{\frac{N-t-1}{N-t}}|Y^{(N)}_{t-1} > x > Y^{(N)}_t] = S^\alpha_t(x).$$

Since e^x is convex, then

$$e^{E[\alpha\Delta^0_{t+1}(Y^{(N)}_t)|Y^{(N)}_{t-1}>x>Y^{(N)}_t]} < E[e^{\alpha\Delta^0_{t+1}(Y^{(N)}_t)}|Y^{(N)}_{t-1} > x > Y^{(N)}_t]$$

and hence

$$e^{E[\alpha\Delta^0_{t+1}(Y^{(N)}_t)|Y^{(N)}_{t-1}>x>Y^{(N)}_t]} < S^\alpha_t(x).$$

Taking logs of both sides of this inequality yields

$$E[\alpha\Delta^0_{t+1}(Y^{(N)}_t)|Y^{(N)}_{t-1} > x > Y^{(N)}_t] < \ln\left(S^\alpha_t(x)\right).$$

Multiplying both sides by $\frac{N-t}{(N-t+1)\alpha}$ yields

$$\begin{aligned}\int_0^x \Delta^0_{t+1}(s)\frac{F(s)^{N-t}f(s)}{F(x)^{N-t+1}}ds &= \frac{K}{N-t+1}E[Y^{(N)}_{N-K}|Y^{(N)}_{t-1} > x > Y^{(N)}_t] \\ &< \frac{N-t}{(N-t+1)\alpha}\ln\left(S^\alpha_t(x)\right).\end{aligned}$$

Adding $-\frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i$ to both sides yields $\delta^0_t(x;\mathbf{p}_{t-1}) < \delta^\alpha_t(x;\mathbf{p}_{t-1})$ for $x > 0$.

We now show that for each $t = 1,\ldots,N-K$ and $\mathbf{p}_{t-1}$ that $\delta^\alpha_t(x;\mathbf{p}_{t-1}) < \gamma_t(x;\mathbf{p}_{t-1})$ for $x > 0$. The proof is by induction. We first show $\delta^\alpha_{N-K}(x;\mathbf{p}_{N-K-1}) < \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$. Since $e^{\alpha s} < e^{\alpha x}$ for $0 < s < x$ then

$$S^\alpha_{N-K}(x) = E[e^{\alpha Y^{(N)}_{N-K}}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}] < e^{\alpha x}.$$

Taking logs of both sides and rearranging yields

$$\frac{K}{(K+1)\alpha}\ln(S^{\alpha}_{N-K}(x)) < \frac{K}{K+1}x.$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$ to both sides yields $\delta^{\alpha}_{N-K}(x;\mathbf{p}_{t-1}) < \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$ for $x > 0$.

Assume for $t \leq N-K-1$ that $\Delta^{\alpha}_{t+1}(x) < Kx/(N-t)$ for $x > 0$. Since $\Delta^{\alpha}_{t+1}(x)$ is increasing, then for $s < x$ we have $\Delta^{\alpha}_{t+1}(s) < \Delta^{\alpha}_{t+1}(x) < Kx/(N-t)$ or $\alpha\Delta^{\alpha}_{t+1}(s) < \alpha\Delta^{\alpha}_{t+1}(x) < \alpha Kx/(N-t)$ and thus

$$e^{\alpha\Delta^{\alpha}_{t+1}(s)} = S^{\alpha}_{t+1}(s)^{\frac{N-t-1}{N-t}} < e^{\alpha\Delta^{\alpha}_{t+1}(x)} < e^{\alpha K\frac{x}{N-t}}.$$

Hence

$$E\left[\left(S^{\alpha}_{t+1}(Y^{(N)}_t)\right)^{\frac{N-t-1}{N-t}}|Y^{(N)}_{t-1} > x > Y^{(N)}_t\right] = S^{\alpha}_t(x) < e^{\alpha K\frac{x}{N-t}}.$$

Taking logs of both sides yields

$$\ln(S^{\alpha}_t(x)) < \alpha K\frac{x}{N-t},$$

and so

$$\frac{N-t}{(N-t+1)\alpha}\ln(S^{\alpha}_t(x)) < \frac{Kx}{N-t+1}.$$

Hence $\Delta^{\alpha}_t(x) < Kx/(N-t+1)$ for $x > 0$. Adding $-\sum_{i=1}^{t-1} p_i$ to each side gives us $\delta^{\alpha}_t(x;\mathbf{p}_{t-1}) < \gamma_t(x;\mathbf{p}_{t-1})$ for $x > 0$. $\square$

Proof of Proposition 6: We establish the results for the chore auction here, and leave the proofs for the goods auction to the Supplemental Appendix. We first show that $\delta^{\alpha}_t(x;\mathbf{p}_{t-1})$ is increasing in α. The proof is by induction. Suppose $\tilde{\alpha} > \alpha$. Since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned}\left(S^{\tilde{\alpha}}_{N-K}(x)\right)^{\frac{\alpha}{\tilde{\alpha}}} &= \left(E[e^{\tilde{\alpha}Y^{(N)}_{N-K}}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}]\right)^{\frac{\alpha}{\tilde{\alpha}}} \\ &> E[\left(e^{\tilde{\alpha}Y^{(N)}_{N-K}}\right)^{\frac{\alpha}{\tilde{\alpha}}}|Y^{(N)}_{N-K-1} > x > Y^{(N)}_{N-K}] \\ &= S^{\alpha}_{N-K}(x).\end{aligned}$$

for $x > 0$. Taking logs and rearranging yields

$$\frac{K}{(K+1)\,\tilde{\alpha}} \ln S_{N-K}^{\tilde{\alpha}}(x) - \frac{1}{K+1} \sum_{i=1}^{N-K-1} p_i > \frac{K}{(K+1)\,\alpha} \ln S_{N-K}^{\alpha}(x) - \frac{1}{K+1} \sum_{i=1}^{N-K-1} p_i.$$

Hence $\delta_{N-K}^{\tilde{\alpha}}(x; \mathbf{p}_{N-K-1}) > \delta_{N-K}^{\alpha}(x; \mathbf{p}_{N-K-1})$.

Let

$$\Delta_{t+1}^{\alpha}(x) = \frac{N-t-1}{(N-t)\alpha} \ln(S_{t+1}^{\alpha}(x)).$$

Suppose $\delta_{t+1}^{\tilde{\alpha}}(x; \mathbf{p}_t) > \delta_{t+1}^{\alpha}(x; \mathbf{p}_t)$ and hence $\Delta_{t+1}^{\tilde{\alpha}}(x) > \Delta_{t+1}^{\alpha}(x)$. We show that $\delta_t^{\tilde{\alpha}}(x; \mathbf{p}_{t-1}) > \delta_t^{\alpha}(x; \mathbf{p}_{t-1})$. Jensen's inequality and $\Delta_{t+1}^{\tilde{\alpha}}(x) > \Delta_{t+1}^{\alpha}(x)$ imply

$$\begin{aligned} \left(E[e^{\tilde{\alpha}\Delta_{t+1}^{\tilde{\alpha}}(Y_t^{(N)})} | Y_{t-1}^{(N)} > x > Y_t^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}} &> E[e^{\alpha\Delta_{t+1}^{\tilde{\alpha}}(Y_t^{(N)})} | Y_{t-1}^{(N)} > x > Y_t^{(N)}] \\ &> E[e^{\alpha\Delta_{t+1}^{\alpha}(Y_t^{(N)})} | Y_{t-1}^{(N)} > x > Y_t^{(N)}]. \end{aligned}$$

Simple algebra yields

$$\begin{aligned} \Delta_t^{\tilde{\alpha}}(x) &= \frac{N-t}{(N-t+1)\tilde{\alpha}} \ln E[e^{\tilde{\alpha}\Delta_{t+1}^{\tilde{\alpha}}(Y_t^{(N)})} | Y_{t-1}^{(N)} > x > Y_t^{(N)}] \\ &> \frac{N-t}{(N-t+1)\alpha} \ln E[e^{\alpha\Delta_{t+1}^{\alpha}(Y_t^{(N)})} | Y_{t-1}^{(N)} > x > Y_t^{(N)}] \\ &= \Delta_t^{\tilde{\alpha}}(x), \end{aligned}$$

and therefore that $\delta_t^{\tilde{\alpha}}(x; \mathbf{p}_{t-1}) > \delta_t^{\alpha}(x; \mathbf{p}_{t-1})$.

Next we prove that $\lim_{\alpha \to \infty} \delta_t^{\alpha}(x; \mathbf{p}_{t-1}) = \gamma_t(x; \mathbf{p}_{t-1})$. The bid function $\delta_t^{\alpha}(x; \mathbf{p}_{t-1})$ can be written as

$$\delta_t^{\alpha}(x; \mathbf{p}_{k-1}) = \frac{1}{\alpha} \ln \left(S_t^{\alpha}(x)^{\frac{N-t}{N-t+1}} \right) - \sum_{i=1}^{t-1} \frac{1}{N-t+1} p_i.$$

By the definition of $S_t^{\alpha}(x)$ and by iteratively applying Jensen's inequality we obtain

$$S_t^{\alpha}(x)^{\frac{N-t}{N-t+1}} \geq E[e^{\frac{\alpha K}{N-t+1} Y_{N-K}^{(N)}} | Y_{t-1}^{(N)} > x > Y_t^{(N)}].$$

Thus we have

$$\frac{1}{\alpha}\ln(S_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}) \geq \frac{1}{\alpha}\ln\left(E[e^{\frac{\alpha K}{N-t+1}Y_{N-K}^{(N)}}|Y_{t-1}^{(N)} > x > Y_t^{(N)}]\right).$$

The round t equilibrium bid function therefore is bounded below by

$$\delta_t^{\alpha}(x;\mathbf{p}_{t-1}) \geq \frac{1}{\alpha}\ln\left(E[e^{\frac{\alpha K}{N-t+1}Y_{N-K}^{(N)}}|Y_{t-1}^{(N)} > x > Y_t^{(N)}]\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By Proposition 5 we have that

$$\gamma_t(x;\mathbf{p}_{t-1}) \geq \delta_t^{\alpha}(x;\mathbf{p}_{t-1}).$$

We complete the proof by establishing that $\lim_{\alpha\to\infty}\frac{1}{\alpha}\ln\left(E[e^{\frac{\alpha K}{N-t+1}Y_{N-K}^{(N)}}|Y_{t-1}^{(N)} > x > Y_t^{(N)}]\right) = \frac{K}{N-t+1}x$, i.e.,

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\ln\left(\int_0^x e^{\frac{\alpha K s}{N-t+1}}g(s)ds\right) = \frac{K}{N-t+1}x,$$

where

$$g(s) = \frac{(N-t+1)!}{(N-K-t)!K!}\frac{F(s)^K\left[F(x)-F(s)\right]^{N-K-t}}{F(x)^{N-t+1}}f(s).$$

The result then follows from the Squeeze Theorem.

We now establish the above limit. Applying l'Hopital's rule, this limit equals

$$\frac{K}{N-t+1}\lim_{\alpha\to\infty}\frac{\int_0^x se^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds}.$$

We establish that that following limit holds for any $0 < g(s) < \infty$.

$$\lim_{\alpha\to\infty}\frac{\int_0^x se^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds} = x$$

First, we have

$$\frac{\int_0^x se^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds} \leq \frac{x\int_0^x e^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds} = x.$$

Second, for $\Delta > 0$ small we may write

$$\int_0^x se^{\alpha s}g(s)ds = \int_{x-\Delta}^x se^{\alpha s}g(s)ds + \int_0^{x-\Delta} se^{\alpha s}g(s)ds$$

so

$$\begin{aligned}
\lim_{\alpha\to\infty}\frac{\int_0^x se^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds} &\geq \lim_{\alpha\to\infty}\frac{(x-\Delta)\int_0^x e^{\alpha s}g(s)ds - \int_0^{x-\Delta}(x-\Delta-s)\,e^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds} \\
&= (x-\Delta) - \lim_{\alpha\to\infty}\frac{\int_0^{x-\Delta}(x-\Delta-s)\,e^{\alpha s}g(s)ds}{\int_0^x e^{\alpha s}g(s)ds} \\
&\geq (x-\Delta) - \lim_{\alpha\to\infty}\frac{e^{\alpha(x-\Delta)}\int_0^{x-\Delta}(x-\Delta-s)\,g(s)ds}{\int_{x-\frac{1}{2}\Delta}^x e^{\alpha s}g(s)ds} \\
&\geq (x-\Delta) - \lim_{\alpha\to\infty}\frac{e^{\alpha(x-\Delta)}\int_0^{x-\Delta}(x-\Delta-s)\,g(s)ds}{e^{\alpha(x-\frac{1}{2}\Delta)}\int_{x-\frac{1}{2}\Delta}^x g(s)ds} \\
&= (x-\Delta) - \frac{\int_0^{x-\Delta}(x-\Delta-s)\,g(s)ds}{\int_{x-\frac{1}{2}\Delta}^x g(s)ds}\lim_{\alpha\to\infty}e^{\alpha(-\frac{1}{2}\Delta)} \\
&= x-\Delta,
\end{aligned}$$

where the last equality follows since $\lim_{\alpha\to\infty} e^{\alpha(-\frac{1}{2}\Delta)} = 0$. This inequality holds for any Δ positive (and small) and hence we have established the desired limit. The main result then follows from the Squeeze Theorem. □

Proof of Proposition 7: We construct γ recursively, showing that the strategy guarantees a payoff at round t in the chore auction of at least

$$\bar{v}_t(x_i, \mathbf{p}_{t-1}) = -\frac{K\left(x_i - \frac{1}{K}\sum_{i=1}^{t-1} p_i\right)}{N-t+1}$$

and guaranteeing at least $-\bar{v}_t(x_i, \mathbf{p}_{t-1})$ in the goods auction.

Consider round $N-K$. In the chore auction, a bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of $-b$, or (ii) a rival drops first at $p_{N-K} \geq b$ and he obtains

$$\frac{1}{K}\sum_{i=1}^{N-K-1} p_i + \frac{1}{K}p_{N-K} - x.$$

The bidder maximizes his minimum payoff when b satisfies

$$-b = \frac{1}{K}\sum_{i=1}^{N-K-1} p_i + \frac{1}{K}b - x,$$

i.e., $b = \gamma_{N-K}(x; \mathbf{p}_{N-K-1})$. Hence at round $N-K$ the bidder guarantees himself a payoff of at least $-\gamma_{N-K}(x; \mathbf{p}_{N-K-1}) = \bar{v}_{N-K}(x; \mathbf{p}_{N-K-1})$.

The argument is the same for the goods auction, except that the signs of the payoffs are reversed. A bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of b, or (ii) a rival drops first at $p_{N-K} \leq b$ and he obtains

$$-\left(\frac{1}{K}\sum_{i=1}^{N-K-1} p_i + \frac{1}{K}p_{N-K} - x\right).$$

Hence the bidder maximizes his minimum payoff when $b = \gamma_{N-K}(x; \mathbf{p}_{N-K-1})$ and he guarantees himself a payoff of at least $\gamma_{N-K}(x; \mathbf{p}_{N-K-1}) = -\bar{v}_{N-K}(x; \mathbf{p}_{N-K-1})$. Since the argument is the same, hereafter we focus on the chore auction.

Suppose that at round $t+1$, given $\mathbf{p}_t$ a bidder with value x can guarantee himself $\bar{v}_{t+1}(x, \mathbf{p}_t)$. Consider round t. A bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of $-b$, or (ii) a rival drops first at $p_t \geq b$ and he obtains at least $\bar{v}_{t+1}(x, \mathbf{p}_t) \geq \bar{v}_{t+1}(x, (\mathbf{p}_{t-1}, b))$. His minimum payoff is maximized when $-b = \bar{v}_{t+1}(x, (\mathbf{p}_{t-1}, b))$, i.e., $b = \gamma_t(x; \mathbf{p}_{t-1})$. He obtains a payoff of at least $-\gamma_t(x; \mathbf{p}_{t-1}) = \bar{v}_t(x; \mathbf{p}_{t-1})$.

Next we show that $\bar{v}_t(x; \mathbf{p}_{t-1})$ is the largest payoff that a bidder can guarantee himself at round t given $\mathbf{p}_{t-1}$. Suppose to the contrary that he can guarantee himself $v'_t > \bar{v}_t(x; \mathbf{p}_{t-1})$. If all active bidders have the same value x then, since the game is symmetric, each bidder can guarantee himself v'_t and hence the total guaranteed payoff of the active bidders is at least

$$(N - t + 1)v'_t > (N - t + 1)\bar{v}_t(x; \mathbf{p}_{t-1}) = \sum_{i=1}^{t-1} p_i - Kx.$$

This is a contradiction since the right hand side is the total surplus that can be obtained by the active bidders at round t: In subsequent rounds, any additional compensation $p_t, \ldots, p_{N-K}$ that is received by a currently active bidder is also paid by a currently active bidder, and hence generates no additional surplus.

The proof that γ is the unique maxmin perfect strategy is straightforward so we only sketch it here. Suppose $\hat{\gamma} \neq \gamma$ is a maxmin perfect strategy. Then there is some x, t, and $\mathbf{p}_{t-1}$ such that $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) \neq \gamma_t(x; \mathbf{p}_{t-1})$. Suppose $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) > \gamma_t(x; \mathbf{p}_{t-1})$. If the bidder drops at round t he obtains a payoff of $-\hat{\gamma}_t(x; \mathbf{p}_{t-1}) < -\gamma_t(x; \mathbf{p}_{t-1}) = \bar{v}_t(x; \mathbf{p}_{t-1})$.

Suppose $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) < \gamma_t(x; \mathbf{p}_{t-1})$ and rival bidder drops out $\hat{p}_t$ such that $\hat{\gamma}_t(x; \mathbf{p}_{t-1}) < \hat{p}_t < \gamma_t(x; \mathbf{p}_{t-1})$. One can show that the other bidders can hold him to a payoff of no more than $\bar{v}_{t+1}(x; (\mathbf{p}_{t-1}, \hat{p}_t))$. We have

$$\bar{v}_{t+1}(x; (\mathbf{p}_{t-1}, \hat{p}_t)) < \bar{v}_{t+1}(x; (\mathbf{p}_{t-1}, \gamma_t(x; \mathbf{p}_{t-1}))) = \bar{v}_t(x; \mathbf{p}_{t-1}),$$

where the inequality holds since $\bar{v}_{t+1}$ is increasing in p_t and the equality holds by construction. $\square$

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7 Supplemental Appendix (not for publication)

This appendix contains proofs for the goods auction.

Proof of Proposition 1′: Let $\beta = (\beta_1, \dots, \beta_{N-K})$ be a symmetric equilibrium in increasing and differentiable strategies. For each $t \leq N - K$, let $\pi_t(\hat{x}, x|\mathbf{z}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is $\hat{x}$ (i.e., he bids $\beta_t(\hat{x}|\mathbf{z}_{t-1})$), when $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop so far. In this case we will sometimes say the bidder "bids $\hat{x}$". Let

$$\Pi_t(x|\mathbf{z}_{t-1}) = \pi_t(x, x|\mathbf{z}_{t-1})$$

be the bidder's equilibrium payoff in round t.

(a) For each $\mathbf{z}_{t-1}$:

- (a.i) β_t satisfies the differential equation given in Proposition 1'(i).
- (a.ii) if $x \geq z_{t-1}$ then $x \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{z}_{t-1})$, i.e., it is optimal for each bidder to follow β_t in round t; if $x < z_{t-1}$ then $z_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{z}_{t-1})$.

(b) For each $\mathbf{z}_{t-1}$:

$$\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} \geq 0\,.$$

We prove by induction that the claim is true for each $t \in \{1, \dots, N-K\}$, thereby establishing Proposition 1. Note that since equilibrium is in increasing strategies, at any round t the sequence of dropout prices $(p_1, \dots, p_{t-1})$ reveals the $t-1$ lowest values $\mathbf{z}_{t-1} = (z_1, \dots, z_{t-1})$.

Let $\mathbf{z}_{N-K-1}$ be arbitrary and consider an active bidder whose value is x but who bids as though it is $\hat{x} \geq z_{N-K-1}$. There are two cases to consider: (i) $x \geq z_{N-K-1}$ and (ii) $x < z_{N-K-1}$.

Case (i): $x \geq z_{N-K-1}$. With a bid of $\hat{x} \geq z_{N-K-1}$, if $\hat{x} > z_{N-K}$ then a rival bidder drops out first at the price $\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1})$, the bidder wins an item, and he receives compensation of

$$x - \frac{1}{K}\left(\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i\right).$$

If $\hat{x} < z_{N-K}$ then the bidder drops before any rival and he obtains compensation $\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})$. Hence $\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1}) =$

$$\int_{z_{N-K-1}}^{\hat{x}} u(x - \tfrac{1}{K}\left(\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i\right))h_{N-K}^{(N-1)}(z_{N-K}|z_{N-K-1})dz_{N-K}$$
$$+\int_{\hat{x}}^{\bar{x}} u(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))h_{N-K}^{(N-1)}(z_{N-K}|z_{N-K-1})dz_{N-K}.$$

Differentiating with respect to $\hat{x}$ yields $\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial\hat{x} =$

$$\begin{aligned} &u'(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))\beta'_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})(1 - H_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1})) \\ &-u(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) \\ &+u\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) \end{aligned} \quad (6)$$

A necessary condition for β to be an equilibrium is that $\partial\pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial\hat{x}|_{\hat{x}=x} = 0$, i.e.,

$$\begin{aligned} &u'(\beta_{N-K}(x|\mathbf{z}_{N-K-1}))\beta'_{N-K}(x|\mathbf{z}_{N-K-1}) \\ &= \left[u(\beta_{N-K}(x|\mathbf{z}_{N-K-1})) - u\left(x - (\tfrac{1}{K}\beta_{N-K}(x|\mathbf{z}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i)\right)\right]\lambda_{N-K}^{N}(x). \end{aligned} \quad (7)$$

where

$$\frac{h_{N-K}^{(N-1)}(x|z_{N-K-1})}{1 - H_{N-K}^{(N-1)}(x|z_{N-K-1})} = \frac{Kf(x)}{1 - F(x)} = \lambda_{N-K}^{N}(x)$$

Alternatively, since types can be inferred from dropout prices, we can write the necessary condition as

$$u'(\beta_{N-K}(x;\mathbf{p}_{N-K-1}))\beta'_{N-K}(x;\mathbf{p}_{N-K-1})$$
$$= \Big[u(\beta_{N-K}(x|\mathbf{p}_{N-K-1})) - ux - \tfrac{1}{K}(\beta_{N-K}(x|\mathbf{p}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i)\Big]]\lambda^N_{N-K}(x)$$

which establishes (a.i) for $t = N - K$.

The necessary condition holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$u'(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}))\beta'_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})$$
$$= \Big[u(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1})) - u\left(\hat{x} - \tfrac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \textstyle\sum_{i=1}^{N-K-1} p_i)\right)\Big]]\lambda^N_{N-K}(\hat{x}) \quad (8)$$

Substituting (8) into (6) and simplifying yields

$$\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial \hat{x}} = \begin{bmatrix} u\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \\ -u\left(\hat{x} - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \end{bmatrix} h^{(N-1)}_{N-K}(\hat{x}|z_{N-K-1}).$$

Clearly, $\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial \hat{x}|_{\hat{x}=x} = 0$. Moreover, for $\hat{x} \geq z_{N-K-1}$ we have

$$\frac{\partial^2 \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial \hat{x} \partial x} = u'\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) h^{(N-1)}_{N-K}(\hat{x}|z_{N-K-1}) \geq 0,$$

where the inequality holds since $u' > 0$ and $h^{(N-1)}_{N-K}(\hat{x}|z_{N-K-1}) \geq 0$. Hence, if $x \geq z_{N-K-1}$ then $x \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})$ by Lemma 0 of McAfee (1992).

Case (ii): $x < z_{N-K-1}$. It is clearly never optimal for a bidder to bid as though his type is less than z_{N-K-1}, i.e., bid less than $\beta_{N-K}(z_{N-K-1}|\mathbf{z}_{N-K-1})$, since he receives more compensation with a bid of $\beta_{N-K}(z_{N-K-1}|\mathbf{z}_{N-K-1})$. (For either bid he drops out for sure since the other bidders have values above $\mathbf{z}_{N-K-1}$.)

For $\hat{x} \geq z_{N-K-1}$ we have

$$\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial \hat{x}} = \left[\begin{array}{c} u\left(x - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \\ -u\left(\hat{x} - \frac{1}{K}(\beta_{N-K}(\hat{x}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) \end{array} \right] h_{N-K}^{(N-1)}(\hat{x}|z_{N-K-1}) < 0$$

and thus $z_{N-K-1} \in \arg\max_{\hat{x}} \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})$. Hence (a.ii) is true for $t = N - K$.

To prove (b), note that $d\Pi_{N-K}(x|\mathbf{z}_{N-K-1})/dx$ is

$$\begin{aligned} & \left.\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial \hat{x}}\right|_{\hat{x}=x} + \left.\frac{\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})}{\partial x}\right|_{\hat{x}=x} \\ = & \int\limits_{z_{N-K-1}}^{x} u'\left(x - \frac{1}{K}(\beta_{N-K}(z_{N-K}|\mathbf{z}_{N-K-1}) + \sum_{i=1}^{N-K-1} p_i)\right) h_{N-K}^{(N-1)}(z_{N-K}|z_{N-K-1}) dz_{N-K} \\ \geq & \ 0, \end{aligned}$$

where the second equality holds since $\partial \pi_{N-K}(\hat{x}, x|\mathbf{z}_{N-K-1})/\partial \hat{x}|_{\hat{x}=x} = 0$. Hence (b) holds for $t = N - K$.

Assume the claim is true for rounds $t + 1$ through $N - 1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be arbitrary. If $x < z_{t-1}$ then, by the same argument as before, $z_{t-1} \in \arg\max_{\hat{x}} \pi_t(\hat{x}, x|\mathbf{z}_{t-1})$.

Suppose $x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $\hat{x} \geq z_{t-1}$. We need to distinguish between two cases: (i) $\hat{x} \in [z_{t-1}, x]$ and (ii) $\hat{x} > x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of $\hat{x}$ as $\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})$ if $\hat{x} \in [z_{t-1}, x]$ and as $\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})$ if $\hat{x} \geq x$.

Case (i): Suppose $\hat{x} \in [z_{t-1}, x]$. If $z_t \in [z_{t-1}, \hat{x}]$ the bidder continues to round $t + 1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq \hat{x}$ he receives compensation of $\beta_t(\hat{x}|\mathbf{z}_{t-1})$. Hence his payoff is

$$\begin{aligned} \pi_t^L(\hat{x}, x|\mathbf{z}_{t-1}) = & \int_{z_{t-1}}^{\hat{x}} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) h_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \int_{\hat{x}}^{\bar{x}} u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(z_t|z_{t-1}) dz_t. \end{aligned}$$

Differentiating with respect to $\hat{x}$ yields $\partial\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})/\partial\hat{x} =$

$$\begin{aligned}
&\Pi_{t+1}(x|\mathbf{z}_{t-1}, \hat{x})h_t^{(N-1)}(\hat{x}|z_{t-1}) - u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right)h_t^{(N-1)}(\hat{x}|z_{t-1}) \\
&+u'\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right)\beta_t'(\hat{x}|\mathbf{z}_{t-1})(1 - H_t^{(N-1)}(\hat{x}|z_{t-1})).
\end{aligned}$$

Since

$$\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) = u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x)),$$

and

$$\frac{h_t^{(N-1)}(x|z_{t-1})}{1 - H_t^{(N-1)}(x|z_{t-1})} = (N-t)\frac{f(x)}{1-F(x)} = \lambda_t^N(x),$$

the necessary condition for equilibrium that $\partial\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})/\partial\hat{x}|_{\hat{x}=x} \geq 0$ can be written as

$$\begin{aligned}
&u'\left(\beta_t(x|\mathbf{z}_{t-1})\right)\beta_t'(x|\mathbf{z}_{t-1}) \\
\geq\ &\left[u\left(\beta_t(x|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))\right]\lambda_t^N(x).
\end{aligned} \tag{9}$$

Also, for $\hat{x} \in [z_{t-1}, x]$ we have

$$\frac{\partial^2\pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})}{\partial\hat{x}\partial x} = \frac{d}{dx}\Pi_{t+1}(x|\mathbf{z}_{t-1}, \hat{x})h_t^{(N-1)}(\hat{x}|z_{t-1}) \geq 0,$$

where the inequality follows since (b) is true for round $t+1$ by the induction hypothesis.

Case (ii): Suppose $\hat{x} \geq x$. If $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, \hat{x}]$, then he continues to round $t+1$ and, by the induction hypothesis, he bids z_t and receives compensation of $\beta_{t+1}(z_t|\mathbf{z}_{t-1}, z_t)$. If $z_t > \hat{x}$ then in round t he receives compensation of $\beta_t(\hat{x}|\mathbf{z}_{t-1})$. His payoff at round t is therefore

$$\begin{aligned}
\pi_t^H(\hat{x}, x|\mathbf{z}_{t-1}) = \quad &\textstyle\int_{z_{t-1}}^{x}\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)h_t^{(N-1)}(z_t|z_{t-1})dz_t \\
&+\textstyle\int_{x}^{\hat{x}}u(\beta_{t+1}(z_t|\mathbf{z}_{t-1}, z_t))h_t^{(N-1)}(z_t|z_{t-1})dz_t, \\
&+\textstyle\int_{\hat{x}}^{\bar{x}}u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right)h_t^{(N-1)}(z_t|z_{t-1})dz_t.
\end{aligned}$$

Differentiating with respect to $\hat{x}$ yields

$$\begin{aligned}\frac{\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}} &= u(\beta_{t+1}(\hat{x}|\mathbf{z}_{t-1}, \hat{x}))h_t^{(N-1)}(\hat{x}|z_{t-1}) - u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(\hat{x}|z_{t-1}) \\ &\quad + u'\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) \beta_t'(\hat{x}|\mathbf{z}_{t-1})(1 - H_t^{(N-1)}(\hat{x}|z_{t-1})).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})/\partial \hat{x}|_{\hat{x}=x} \leq 0$, i.e.,

$$\begin{aligned}& u'\left(\beta_t(x|\mathbf{z}_{t-1})\right) \beta_t'(x|\mathbf{z}_{t-1}) \\ \leq\; & [u\left(\beta_t(x|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))]\lambda_t^N(x).\end{aligned} \tag{10}$$

Equations (9) and (10) imply that

$$\begin{aligned}& u'\left(\beta_t(x|\mathbf{z}_{t-1})\right) \beta_t'(x|\mathbf{z}_{t-1}) \\ =\; & [u\left(\beta_t(x|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))]\lambda_t^N(x).\end{aligned} \tag{11}$$

Since the bid functions are increasing, we can replace $\mathbf{z}_{t-1}$ with $\mathbf{p}_{t-1}$ and replace $\beta_{t+1}(x|\mathbf{z}_{t-1}, x)$ with $\beta_{t+1}(x; \mathbf{p}_{t-1}, \beta_t(x; \mathbf{p}_{t-1}))$, writing the first order condition as

$$\begin{aligned}& u'\left(\beta_t(x; \mathbf{p}_{t-1})\right) \beta_t'(x; \mathbf{p}_{t-1}) \\ =\; & [u\left(\beta_t(x; \mathbf{p}_{t-1})\right) - u(\beta_{t+1}(x; \mathbf{p}_{t-1}, \beta_t(x; \mathbf{p}_{t-1})))]\lambda_t^N(x),\end{aligned}$$

which establishes (a.i) for round t.

Equation (11) holds for all x and, in particular, it holds for $x = \hat{x}$, i.e.,

$$\begin{aligned}& u'\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) \beta_t'(\hat{x}|\mathbf{z}_{t-1}) \\ =\; & [u\left(\beta_t(\hat{x}|\mathbf{z}_{t-1})\right) - u(\beta_{t+1}(\hat{x}|\mathbf{z}_{t-1}, \hat{x}))]\lambda_t^N(\hat{x}).\end{aligned}$$

Substituting this expression into the expression for $\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})/\partial \hat{x}$ yields

$$\frac{\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}} = 0 \text{ for } \hat{x} \geq x.$$

Furthermore,

$$\frac{\partial^2 \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x} \partial x} = 0 \text{ for } \hat{x} \geq x.$$

We have shown that

$$\left.\frac{\partial \pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}}\right|_{\hat{x}=x} = \left.\frac{\partial \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}}\right|_{\hat{x}=x} = 0$$

and

$$\frac{\partial^2 \pi_t^L(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}\partial x} \geq 0 \text{ for } \hat{x} \in [z_{t-1}, x] \text{ and } \frac{\partial^2 \pi_t^H(\hat{x}, x|\mathbf{z}_{t-1})}{\partial \hat{x}\partial x} \geq 0 \text{ for } \hat{x} \geq x.$$

Hence (a.ii) is true for round t by Van Essen and Wooders' (2016) extension of McAfee's (1992) Lemma 0.

To establish (b) is true for round t, observe that

$$\begin{aligned} \Pi_t(x|\mathbf{z}_{t-1}) &= \int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) h_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ &\quad + \int_{x}^{\bar{x}} u\left(\beta_t(x|\mathbf{z}_{t-1})\right) h_t^{(N-1)}(z_t|z_{t-1}) dz_t. \end{aligned}$$

Differentiating and simplifying yields

$$\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} = \int_{z_{t-1}}^{x} \frac{d}{dx}\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) h_t^{(N-1)}(z_t|z_{t-1}) dz_t \geq 0,$$

where the equality follows from $\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) = u(\beta_{t+1}(x|\mathbf{z}_{t-1}, x))$ and (11), and the inequality follows since $d\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)/dx \geq 0$ by the induction hypothesis. □

Proof of Proposition 2.2: The proof is by induction. By Proposition $1'$(i), at round $N-K$ the differential equation for the equilibrium bid function is

$$\beta'_{N-K}(x|\mathbf{p}_{N-K-1}) = \left[\frac{K+1}{K}\beta_{N-K}(x|\mathbf{p}_{N-K-1}) + \frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right]\lambda_{N-K}^N(x).$$

Multiplying both sides by $1 - F(x)$ we obtain

$$\beta'_{N-K}(x|\mathbf{p}_{N-K-1})\left(1 - F(x)\right) - (K+1)\beta_{N-K}(x|\mathbf{p}_{N-K-1})f(x) = \left[\frac{1}{K}\sum_{i=1}^{N-K-1} p_i - x\right] Kf(x)$$

i.e.,

$$\frac{d}{dx}\left(\beta_{N-K}(x|\mathbf{p}_{N-K-1})\left(1-F(x)\right)^{K+1}\right)=\left[\frac{1}{K}\sum\nolimits_{i=1}^{N-K-1}p_i-x\right]Kf(x)\left(1-F(x)\right)^{K}.$$

By the Fundamental Theorem of Calculus we have

$$\beta_{N-K}(x|\mathbf{p}_{N-K-1})\left(1-F(x)\right)^{K+1}=-\int_0^x[s-\frac{1}{K}\sum\nolimits_{i=1}^{N-K-1}p_i]Kf(s)\left(1-F(s)\right)^{K}ds+C.$$

The LHS of this equation is zero when $x=\bar{x}$ (since $F(\bar{x})=1$), which implies

$$C=\int_0^{\bar{x}}[s-\frac{1}{K}\sum\nolimits_{i=1}^{N-K-1}p_i]Kf(s)\left(1-F(s)\right)^{K}ds.$$

Since

$$\int_x^{\bar{x}}s(K+1)\frac{\left(1-F(s)\right)^{K}}{\left(1-F(x)\right)^{K+1}}f(s)ds=E\left[Z_{N-K}^{(N)}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}\right],$$

then

$$\beta_{N-K}(x|\mathbf{p}_{N-K-1})=\frac{K}{K+1}E\left[Z_{N-K}^{(N)}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}\right]-\frac{1}{K+1}\sum\nolimits_{i=1}^{N-K-1}p_i$$

which establishes the result for round $N-K$.

Assume in round t that

$$\beta_t(x;\mathbf{p}_{t-1})=\frac{K}{N-t+1}E\left[Z_{N-K}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]-\frac{1}{N-t+1}\sum_{i=1}^{t-1}p_i.$$

We need to show that $\beta_{t-1}(x;\mathbf{p}_{t-2})$ is as given in Proposition 2. The differential equation for round $t-1$ is

$$\beta_{t-1}'(x;\mathbf{p}_{t-2})=\left[\beta_{t-1}(x|\mathbf{p}_{t-2})-\beta_t(x|\mathbf{p}_{t-2},\beta_{t-1}(x|\mathbf{p}_{t-2}))\right]\lambda_{t-1}^{N}(x).$$

By the induction hypothesis

$$\begin{aligned}\beta_t(x;\mathbf{p}_{t-2},\beta_{t-1}(x|\mathbf{p}_{t-2})) &= \frac{K}{N-t+1}E\left[Z_{N-K}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right]\\ &\quad-\frac{1}{N-t+1}\left(\sum_{i=1}^{t-2}p_i+\beta_{t-1}(x|\mathbf{p}_{t-2})\right).\end{aligned}$$

Hence,

$$\beta'_{t-1}(x;\mathbf{p}_{t-2}) = \left[\begin{array}{c} \frac{N-t+2}{N-t+1}\beta_{t-1}(x|\mathbf{p}_{t-2}) - \frac{K}{N-t+1}E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t} > x > Z^{(N)}_{t-1}\right] \\ +\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i \end{array}\right]\lambda^{N}_{t-1}(x).$$

Multiplying both sides by $(1-F(x))^{N-t+2}$ yields $\frac{d}{dx}\left(\beta_{t-1}(x|\mathbf{p}_{t-2})\,(1-F(x))^{N-t+2}\right) =$

$$\left[\frac{\sum_{i=1}^{t-2}p_i}{N-t+1} - KE\left[Z^{(N)}_{N-K}|Z^{(N)}_{t} > x > Z^{(N)}_{t-1}\right]\right] f(x)\,(1-F(x))^{N-t+1}.$$

By the Fundamental Theorem of Calculus and since $F(\bar{x}) = 1$ then

$$\beta_{t-1}(x|\mathbf{p}_{t-2}) = \int_{x}^{\bar{x}}\left[\frac{\sum_{i=1}^{t-2}p_i}{N-t+1} - KE\left[Z^{(N)}_{N-K}|Z^{(N)}_{t} > s > Z^{(N)}_{t-1}\right]\right] f(s)\frac{(1-F(s))^{N-t+1}}{(1-F(x))^{N-t+2}}ds.$$

Since (to be established momentarily)

$$\begin{aligned} &\int_{x}^{\bar{x}} E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t} > s > Z^{(N)}_{t-1}\right] f(s)\frac{(N-t+2)\,(1-F(s))^{N-t+1}}{(1-F(x))^{N-t+2}}ds \\ =\;& E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t-1} > x > Z^{(N)}_{t-2}\right] \end{aligned}$$

then

$$\beta'_{t-1}(x;\mathbf{p}_{t-2}) = \frac{K}{N-t+2}\left(E\left[Z^{(N)}_{N-K}|Z^{(N)}_{t-1} > x > Z^{(N)}_{t-2}\right] - \frac{1}{K}\sum_{i=1}^{t-2}p_i\right)$$

which completes the proof.

Finally we establish the equality we just used. We have

$$\begin{aligned}
&\int_x^{\bar{x}} E\left[Z_{N-K}^{(N)}|Z_t^{(N)} > s > Z_{t-1}^{(N)}\right] f(s)\frac{(N-t+2)\left(1-F(s)\right)^{N-t+1}}{\left(1-F(x)\right)^{N-t+2}}ds \\
= &\int_x^{\bar{x}} \left(\int_s^{\bar{x}} \frac{r(N-t+1)!\left[F(r)-F(s)\right]^{N-K-t}\left[1-F(r)\right]^K f(r)dr}{(N-K-t)!K!\left[1-F(s)\right]^{N-t+1}}\right) \\
&\times\frac{(N-t+2)f(s)\left(1-F(s)\right)^{N-t+1}}{\left(1-F(x)\right)^{N-t+2}}ds \\
= &\int_x^{\bar{x}} \left(\int_x^r r\frac{(N-t+1)!}{(N-K-t)!K!}\frac{\left[F(r)-F(s)\right]^{N-K-t}\left[1-F(r)\right]^K}{\left(1-F(x)\right)^{N-t+2}}\right)(N-t+2)f(r)f(s)dsdr \\
= &\int_x^{\bar{x}} r\frac{(N-t+2)!}{(N-K-t+1)!K!}\frac{\left[F(r)-F(s)\right]^{N-K-t+1}\left[1-F(r)\right]^K}{\left(1-F(x)\right)^{N-t+2}}f(r)dr \\
&E\left[Z_{N-K}^{(N)}|Z_{t-1}^{(N)} > x > Z_{t-2}^{(N)}\right].\square
\end{aligned}$$

Proof of Proposition 4.2: To save space we write β_t rather than $\beta_t(x;\mathbf{p}_{t-1})$. At round $t = N-K$, the differential equation that characterizes equilibrium behavior is

$$\frac{d}{dx}\left(e^{-\alpha(\frac{K+1}{K}\beta_{N-K})}\left(1-F(x)\right)^{K+1}\right) = -\left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)e^{-\alpha x}\left(K+1\right)f(x)\left(1-F(x)\right)^K.$$

From the Fundamental Theorem of Calculus, $e^{-\alpha(\frac{K+1}{K}\beta_{N-K})}\left(1-F(x)\right)^{K+1} =$

$$-\left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)\int_0^x e^{-\alpha z}\left(K+1\right)f(z)\left(1-F(z)\right)^K dz + C$$

At $x = \bar{x}$, the LHS of the above equation is equal to zero and hence

$$C = \left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)\int_0^{\bar{x}} e^{-\alpha z}\left(K+1\right)f(z)\left(1-F(z)\right)^K dz.$$

So

$$e^{-\alpha(\frac{K+1}{K}\beta_{N-K})} = \left(e^{\frac{\alpha}{K}\sum_{i=1}^{N-K-1}p_i}\right)\frac{\int_x^{\bar{x}} e^{-\alpha z}\left(K+1\right)f(s)\left(1-F(s)\right)^K ds}{\left(1-F(x)\right)^{K+1}}.$$

Taking logs of both sides we have

$$-\alpha\frac{K+1}{K}\beta_{N-K}=\ln\left(\frac{\int_x^{\bar{x}}e^{-\alpha z}\left(K+1\right)f(s)\left(1-F(s)\right)^K ds}{\left(1-F(x)\right)^{K+1}}\right)+\alpha\frac{1}{K}\sum\nolimits_{i=1}^{N-K-1}p_i,$$

and hence

$$\begin{aligned}\beta_{N-K}^{\alpha}(x;\mathbf{p}_{N-K-1}) &= -\frac{K}{\alpha(K+1)}\ln\left(\frac{\int_x^{\bar{x}}e^{-\alpha z}\left(K+1\right)f(z)\left(1-F(z)\right)^K dz}{\left(1-F(x)\right)^{K+1}}\right)\\ &\quad -\frac{1}{K+1}\sum\nolimits_{i=1}^{N-K-1}p_i.\\ &= -\frac{K}{\alpha(K+1)}\ln\left(E\left[e^{-\alpha Z_{N-K}^{(N)}}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}\right]\right)\\ &\quad -\frac{1}{K+1}\sum\nolimits_{i=1}^{N-K-1}p_i\\ &= -\frac{K}{\alpha(K+1)}\ln\left(D_{N-K}^{\alpha}\left(x\right)\right)-\frac{1}{K+1}\sum\nolimits_{i=1}^{N-K-1}p_i.\end{aligned}$$

Next, we solve for the round $t-1$ bid function. Assume that in round $t\leq N-K$, bidders follow the bid function

$$\beta_t^{\alpha}(x;\mathbf{p}_{t-1})=-\frac{N-t}{\left(N-t+1\right)\alpha}\ln\left(D_t^{\alpha}(x)\right)-\frac{1}{N-t+1}\sum_{i=1}^{t-1}p_i.$$

Note that this implies that $\beta_t^{\alpha}(x;\mathbf{p}_{t-2},\beta_{t-1}^{\alpha}(x;\mathbf{p}_{t-2}))=$

$$-\frac{N-t}{\left(N-t+1\right)\alpha}\ln\left(D_t^{\alpha}(x)\right)-\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i-\frac{1}{N-t+1}\beta_{t-1}^{\alpha}(x;\mathbf{p}_{t-2}).$$

After some manipulation, the differential equation for round $t-1$ from Proposition $1'$ can be written as

$$\begin{aligned}&\frac{d}{dx}\left(e^{-\alpha\frac{N-t+2}{N-t+1}\beta_{t-1}}\left(1-F(x)\right)^{N-(t-1)+1}\right)\\ =\ &-e^{\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}D_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}\left(N-t+2\right)\left(1-F(x)\right)^{N-t+1}f(x).\end{aligned}$$

From the Fundamental Theorem of Calculus we have $e^{-\alpha\frac{N-t+2}{N-t+1}\beta_{t-1}}\left(1-F(x)\right)^{N-(t-1)+1}=$

$$-\int_0^x e^{\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}D_t^\alpha(s)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(s)\right)^{N-t+1}f(s)ds+C.$$

At $x=\bar{x}$, the LHS of the above equation is equal to zero and hence

$$C=\int_0^{\bar{x}} e^{\alpha\left(\frac{1}{N-t+1}\sum_{i=1}^{t-2}p_i\right)}D_t^\alpha(s)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(s)\right)^{N-t+1}f(s)ds.$$

Rearranging yields $\beta_{t-1}(x;\mathbf{p}_{t-2})=$

$$\begin{aligned} & -\frac{1}{N-t+2}\sum_{i=1}^{t-2}p_i-\frac{N-t+1}{(N-t+2)\,\alpha}\ln\left[\frac{\int_x^{\bar{x}}D_t^\alpha(z)^{\frac{N-t}{N-t+1}}(N-t+2)\left(1-F(z)\right)^{N-t+1}f(z)dz}{\left(1-F(x)\right)^{N-t+2}}\right] \\ = \quad & -\frac{1}{N-t+2}\sum_{i=1}^{t-2}p_i-\frac{N-t+1}{(N-t+2)\,\alpha}\ln(D_{t-1}^\alpha(x)) \end{aligned}$$

where the second equality holds since

$$D_{t-1}^\alpha(x)=E\left[\left(D_t^\alpha(Z_{t-1}^{(N)})\right)^{\frac{N-t}{N-t+1}}|Z_{t-1}^{(N)}>x>Z_{t-2}^{(N)}\right].\square$$

Proof of Proposition 5: Here we establish the inequalities for the goods auction.

We show that for $t=1,\ldots,N-K$ and $\mathbf{p}_{t-1}$ that $\beta_t^0(x;\mathbf{p}_{t-1})>\beta_t^\alpha(x;\mathbf{p}_{t-1})$ for $x<\bar{x}$. The proof is by induction. For $t=N-K$, since e^x is a convex function, then by Jensen's Inequality, for $x<\bar{x}$ we have

$$e^{E[-\alpha Z_{N-K}^{(N)}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}]}<E[e^{-\alpha Z_{N-K}^{(N)}}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}].$$

Noting that the RHS is $D_{N-K}^\alpha(x)$, taking the log of both sides, and then multiplying through by $-K/((K+1)\,\alpha)$ yields

$$\frac{K}{K+1}E[Z_{N-K}^{(N)}|Z_{N-K}^{(N)}>x>Z_{N-K-1}^{(N)}]>-\frac{K}{(K+1)\alpha}\ln(D_{N-K}^\alpha(x))$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$ to both sides yields $\beta^0_{N-K}(x;\mathbf{p}_{N-K-1}) > \beta^\alpha_{N-1}(x;\mathbf{p}_{N-K-1})$ for $x < \bar{x}$.

For $t \leq N-K$, define

$$\Sigma^0_t(x) = \frac{K}{N-t+1} E[Z^{(N)}_{N-K} | Z^{(N)}_t > x > Z^{(N)}_{t-1}],$$

and

$$\Sigma^\alpha_t(x) = -\frac{1}{\alpha}\ln\left(D^\alpha_t(x)^{\frac{N-t}{N-t+1}}\right),$$

where $D^\alpha_t(x)$ is defined in Proposition P4.2. We have that

$$e^{-\alpha\Sigma^\alpha_t(x)} = D^\alpha_t(x)^{\frac{N-t}{N-t+1}}.$$

We established above that $\Sigma^0_{N-K}(x) > \Sigma^\alpha_{N-K}(x)$.

Assume for $t \leq N-K-1$ that $\Sigma^0_{t+1}(x) > \Sigma^\alpha_{t+1}(x)$ for $x < \bar{x}$. We show that $\Sigma^0_t(x) > \Sigma^\alpha_t(x)$ for $x < \bar{x}$. Since $-\alpha\Sigma^0_{t+1}(x) < -\alpha\Sigma^\alpha_{t+1}(x)$ and e^x is increasing, then

$$e^{-\alpha\Sigma^0_{t+1}(x)} < e^{-\alpha\Sigma^\alpha_{t+1}(x)} \text{ for } x < \bar{x},$$

or

$$e^{-\alpha\Sigma^0_{t+1}(x)} < D^\alpha_{t+1}(x)^{\frac{N-t-1}{N-t}} \text{ for } x < \bar{x},$$

Thus

$$E[e^{-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)} | Z^{(N)}_t > x > Z^{(N)}_{t-1}] < E[D^\alpha_{t+1}(Z^{(N)}_t)^{\frac{N-t-1}{N-t}} | Z^{(N)}_t > x > Z^{(N)}_{t-1}] = D^\alpha_t(x).$$

Since e^x is convex, then

$$e^{E[-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)|Z^{(N)}_t > x > Z^{(N)}_{t-1}]} < E[e^{-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)} | Z^{(N)}_t > x > Z^{(N)}_{t-1}].$$

and hence

$$e^{E[-\alpha\Sigma^0_{t+1}(Z^{(N)}_t)|Z^{(N)}_t > x > Z^{(N)}_{t-1}]} < D^\alpha_t(x).$$

Taking logs of both sides of this inequality yields

$$E[-\alpha\Sigma^0_{t+1}(Z^{(N)}_t) | Z^{(N)}_t > x > Z^{(N)}_{t-1}] < \ln\left(D^\alpha_t(x)\right).$$

Multiplying both sides by $-\frac{N-t}{(N-t+1)\alpha}$ yields

$$\begin{aligned}\int_x^{\bar{x}} \Sigma^0_{t+1}(z)\frac{(N-t)[1-F(z)]^{N-t}f(z)}{(1-F(x))^{N-t+1}}dz &= \frac{K}{N-t+1}E\left[Z^{(N)}_{N-K}|Z^{(N)}_t > x > Z^{(N)}_{t-1}\right].\\ &> -\frac{N-t}{(N-t+1)\alpha}\ln\left(D^\alpha_t(x)\right).\end{aligned}$$

Adding $-\frac{1}{N-t+1}\sum_{i=1}^{t-1} p_i$ to both sides yields $\beta^0_t(x;\mathbf{p}_{t-1}) > \beta^\alpha_t(x;\mathbf{p}_{t-1})$ for $x < \bar{x}$.

We now show that for each $t = 1,\ldots,N-K$ and $\mathbf{p}_{t-1}$ that $\beta^\alpha_t(x;\mathbf{p}_{t-1}) > \gamma_t(x;\mathbf{p}_{t-1})$ for $x < \bar{x}$. The proof is by induction. We first show $\beta^\alpha_{N-K}(x;\mathbf{p}_{N-K-1}) > \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$.

Since $e^{-\alpha s} < e^{-\alpha x}$ for $x < s < \bar{x}$ then

$$D^\alpha_{N-K}(x) = E[e^{-\alpha Z^{(N)}_{N-K}}|Z^{(N)}_{N-K} > x > Z^{(N)}_{N-K-1}] < e^{-\alpha x}.$$

Taking logs of both sides and rearranging yields

$$-\frac{K}{(K+1)\alpha}\ln(D^\alpha_{N-K}(x)) > \frac{K}{K+1}x,$$

Adding $-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i$ to both sides yields $\beta^\alpha_{N-K}(x;\mathbf{p}_{t-1}) > \gamma_{N-K}(x;\mathbf{p}_{N-K-1})$ for $x < \bar{x}$.

Assume for $t \leq N-K-1$ that $\Sigma^\alpha_{t+1}(x) > Kx/(N-t)$ for $x < \bar{x}$. Since $\Sigma^\alpha_{t+1}(x)$ is increasing, then for $s > x$ we have $\Sigma^\alpha_{t+1}(s) > \Sigma^\alpha_{t+1}(x) > Kx/(N-t)$ or $-\alpha\Sigma^\alpha_{t+1}(s) < -\alpha\Sigma^\alpha_{t+1}(x) < -\alpha Kx/(N-t)$ and thus

$$e^{-\alpha\Sigma^\alpha_{t+1}(s)} = D^\alpha_{t+1}(s)^{\frac{N-t-1}{N-t}} < e^{-\alpha\Sigma^\alpha_{t+1}(x)} < e^{-\alpha K\frac{x}{N-t}}.$$

Hence

$$E[D^\alpha_{t+1}(Z^{(N)}_t)^{\frac{N-t-1}{N-t}}|Z^{(N)}_t > x > Z^{(N)}_{t-1}] = D^\alpha_t(x) < e^{-\alpha K\frac{x}{N-t}}.$$

Taking logs of both sides yields

$$\ln(D^\alpha_t(x)) < -\alpha K\frac{x}{N-t},$$

and so

$$-\frac{N-t}{(N-t+1)\alpha}\ln(D_t^\alpha(x)) > \frac{Kx}{N-t+1}.$$

Hence $\Sigma_t^\alpha(x) > Kx/(N-t+1)$ for $x < \bar{x}$. Adding $-\sum_{i=1}^{t-1} p_i$ to each side gives us

$$\beta_t^\alpha(x;\mathbf{p}_{t-1}) > \gamma_t(x;\mathbf{p}_{t-1}) \text{ for } x < \bar{x}. \ \square$$

Proof of Proposition 6: We first show that $\beta_t^\alpha(x;\mathbf{p}_{t-1})$ is decreasing in α. The proof is by induction. Suppose $\tilde{\alpha} > \alpha$. Since the transformation $y = x^{\frac{\alpha}{\tilde{\alpha}}}$ is concave, then by Jensen's inequality we have that

$$\begin{aligned}
\left(D_{N-K}^{\tilde{\alpha}}(x)\right)^{\frac{\alpha}{\tilde{\alpha}}} &= \left(E[e^{-\tilde{\alpha}Z_{N-K}^{(N)}}|Z_{N-K}^{(N)} > x > Z_{N-K-1}^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}} \\
&> E[\left(e^{-\tilde{\alpha}Z_{N-K}^{(N)}}\right)^{\frac{\alpha}{\tilde{\alpha}}}|Z_{N-K}^{(N)} > x > Z_{N-K-1}^{(N)}] \\
&= D_{N-K}^\alpha(x)
\end{aligned}$$

for $x < \bar{x}$. Taking logs and rearranging yields

$$-\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i - \frac{K}{(K+1)\alpha}\ln D_{N-K}^\alpha(x) > -\frac{1}{K+1}\sum_{i=1}^{N-K-1} p_i - \frac{K}{(K+1)\tilde{\alpha}}\ln D_{N-K}^{\tilde{\alpha}}(x).$$

Hence, $\beta_{N-K}^\alpha(x;\mathbf{p}_{N-K-1}) > \beta_{N-K}^{\tilde{\alpha}}(x;\mathbf{p}_{N-K-1})$.

Let

$$\Sigma_{t+1}^\alpha(x) = -\frac{1}{\alpha}\ln\left(D_t^\alpha(x)^{\frac{N-t-1}{N-t}}\right).$$

Suppose $\beta_{t+1}^\alpha(x;\mathbf{p}_t) > \beta_{t+1}^{\tilde{\alpha}}(x;\mathbf{p}_t)$ and hence $\Sigma_{t+1}^\alpha(x) > \Sigma_{t+1}^{\tilde{\alpha}}(x)$. We show that $\beta_t^\alpha(x;\mathbf{p}_{t-1}) > \beta_t^{\tilde{\alpha}}(x;\mathbf{p}_{t-1})$. Jensen's inequality and $\Sigma_{t+1}^\alpha(x) > \Sigma_{t+1}^{\tilde{\alpha}}(x)$ imply

$$\begin{aligned}
\left(E[e^{-\tilde{\alpha}\Sigma_{t+1}^{\tilde{\alpha}}(Z_t^{(N)})}|Z_t^{(N)} > x > Z_{t-1}^{(N)}]\right)^{\frac{\alpha}{\tilde{\alpha}}} &> E[e^{-\alpha\Sigma_{t+1}^{\tilde{\alpha}}(Z_t^{(N)})}|Z_t^{(N)} > x > Z_{t-1}^{(N)}] \\
&> E[e^{-\alpha\Sigma_{t+1}^{\alpha}(Z_t^{(N)})}|Z_t^{(N)} > x > Z_{t-1}^{(N)}].
\end{aligned}$$

Simple algebra yields

$$\begin{aligned}
\Sigma_t^{\alpha}(x) &= -\frac{N-t}{(N-t+1)\alpha}\ln E[e^{-\alpha\Sigma_{t+1}^{\alpha}(Z_t^{(N)})}|Z_t^{(N)} > x > Z_{t-1}^{(N)}] \\
&> -\frac{N-t}{(N-t+1)\tilde{\alpha}}\ln E[e^{-\tilde{\alpha}\Sigma_{t+1}^{\tilde{\alpha}}(Z_t^{(N)})}|Z_t^{(N)} > x > Z_{t-1}^{(N)}] \\
&= \Sigma_t^{\tilde{\alpha}}(x)
\end{aligned}$$

and therefore that $\beta_t^{\alpha}(x;\mathbf{p}_{t-1}) > \beta_t^{\tilde{\alpha}}(x;\mathbf{p}_{t-1})$.

Next we prove that the $\lim_{\alpha\to\infty}\beta_t^{\alpha}(x;\mathbf{p}_{t-1}) = \gamma_t(x;\mathbf{p}_{t-1})$. The bid function $\beta_t^{\alpha}(x;\mathbf{p}_{t-1})$ can be written as

$$\beta_t^{\alpha}(x;\mathbf{p}_{t-1}) = -\frac{1}{\alpha}\ln\left(D_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By the definition of $D_t^{\alpha}(x)$ and iteratively applying Jensen's Inequality we obtain

Likewise, since $y^{\frac{N-t-1}{N-t+1}}$ is concave, repeating the same argument yields

$$D_t^{\alpha}(x)^{\frac{N-t}{N-t+1}} \geq E[e^{-\frac{\alpha K}{N-t+1}Z_{N-K}^{(N)}}|Z_t^{(N)} > x > Z_{t-1}^{(N)}]. \tag{12}$$

Thus we have

$$\frac{1}{\alpha}\ln(D_t^{\alpha}(x)^{\frac{N-t}{N-t+1}}) \geq \frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha K}{N-t+1}Z_{N-K}^{(N)}}|Z_t^{(N)} > x > Z_{t-1}^{(N)}]\right).$$

The round t equilibrium bid function therefore is bounded above by

$$\beta_t^{\alpha}(x;\mathbf{p}_{t-1}) \leq -\frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha K}{N-t+1}Z_{N-K}^{(N)}}|Z_t^{(N)} > x > Z_{t-1}^{(N)}]\right) - \sum_{i=1}^{t-1}\frac{1}{N-t+1}p_i.$$

By Proposition 5 we have that

$$\gamma_t(x;\mathbf{p}_{t-1}) \leq \beta_t^{\alpha}(x;\mathbf{p}_{t-1}).$$

We complete the proof by establishing that $\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln\left(E[e^{-\frac{\alpha K}{N-t+1}Z_{N-K}^{(N)}}|Z_t^{(N)} > x > Z_{t-1}^{(N)}]\right) = \frac{xK}{N-t+1}$, i.e.,

$$\lim_{\alpha\to\infty} -\frac{1}{\alpha}\ln\left(\int_x^{\bar{x}} e^{-\frac{\alpha K z}{N-t+1}}h(s)ds\right) = \frac{Kx}{N-t+1},$$

where

$$h(s) = \frac{(N-t+1)!}{(N-K-t)!K!} \frac{[F(s)-F(x)]^{N-K-t}(1-F(s))^{K}}{[1-F(x)]^{N-t+1}} f(s).$$

The result then follows from the squeeze theorem.

We now establish the above limit. Applying l'Hopital's rule, this limit equals

$$\lim_{\alpha\to\infty} \frac{K}{N-t+1} \frac{\int_{x}^{\bar{x}} z e^{-\frac{\alpha K z}{N-t+1}} h(z) dz}{\int_{x}^{\bar{x}} e^{-\frac{\alpha K z}{N-t+1}} h(z) dz}.$$

Setting $\tilde{\alpha} = \frac{\alpha K}{N-t+1}$ the desired result is equivalent to showing that

$$\lim_{\tilde{\alpha}\to\infty} \frac{\int_{x}^{\bar{x}} z e^{-\tilde{\alpha} z} h(z) dz}{\int_{x}^{\bar{x}} e^{-\tilde{\alpha} t} h(z) dz} = x$$

This was demonstrated in the proof of Proposition 6 of Van Essen and Wooders (2016). Hence, we have that $\lim_{\alpha\to\infty} \beta_t^{\alpha}(x; \mathbf{p}_{t-1}) = -\sum_{i=1}^{t-1} \frac{1}{N-t+1} p_i + \frac{xK}{N-t+1}$. $\square$