

String bordism invariants in dimension 3 from $U(1)$ -valued TQFTs

(based on joint work with Eugenio Landi, arXiv:2209.12933)

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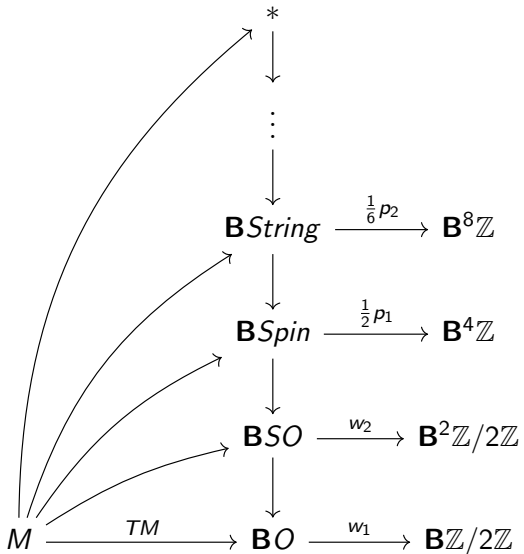
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M n -dimensional smooth manifold

In particular if $n \leq 7$ there are no obstruction to lifting a string structure to a framing.

$$\text{Bord}_3^{\text{String}} = \text{Bord}_3^{\text{fr}} = \pi_3(\mathbb{S}) = \lim_{n \rightarrow +\infty} \pi_{n+3}(S^n) = \pi_8(S^5) = \mathbb{Z}/24\mathbb{Z}$$

↑
(Pontryagin – Thom)

One may wish to express the isomorphism

$\varphi: \text{Bord}_3^{\text{String}} \xrightarrow{\cong} \mathbb{Z}/24\mathbb{Z}$ as some characteristic number given by integrating some *canonical* differential 3-form on a closed string 3-manifold M

$$\varphi[M] = \int_M \omega_M$$

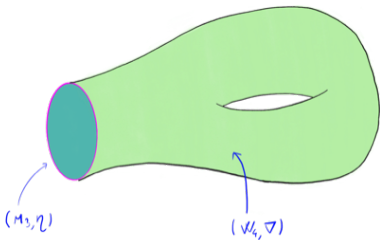
Clearly, there is no hope that this can be true, since the integral takes real values while φ takes values in $\mathbb{Z}/24\mathbb{Z}$, and there is no injective group homomorphism from $\mathbb{Z}/24\mathbb{Z}$ to \mathbb{R} .

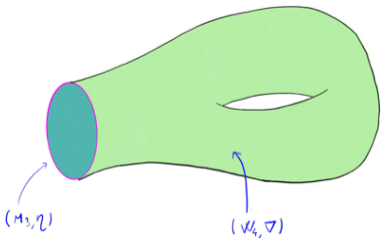
There is however a variant of this construction that may work. Instead of considering just a string 3-manifold M , one considers a string 3-manifold M endowed with some additional structure Υ . This structure should be such that any M admits at least one Υ . To the pair (M, Υ) there could be associated canonical 3-form $\omega_{M, \Upsilon}$ such that $\int_M \omega_{M, \Upsilon}$ takes integral values. Then, if a change in the additional structure Υ results in a change in the value $\int_M \omega_{M, \Upsilon}$ by a multiple of 24 one would have a well defined element

$$\int_M \omega_{M, \Upsilon} \pmod{24}$$

in $\mathbb{Z}/24\mathbb{Z}$, depending only on the string 3-manifold M ; and this could indeed represent the isomorphism φ .

In this form the statement is indeed almost true. The correct version of it has been found by Bunke–Naumann and Redden. Their additional datum Υ consists of a triple (η, W, ∇) , where η is a geometric string structure on M in the sense of Waldorf, W is a spin 4-manifold with $\partial W = M$ and ∇ is a spin connection on W such that the restriction $\nabla|_M$ coincides with the spin connection datum of the geometric string structure η .





$$\psi(M, \eta, W, \nabla) := \frac{1}{2} \int_W \mathbf{p}_1^{CW}(\nabla) - \int_M \omega_\eta,$$

From the interplay between geometric string structures and differential cohomology it follows that $\psi(M, \eta, W, \nabla) \in \mathbb{Z}$. Keeping (M, η) fixed and letting (W, ∇) vary, one finds

$$\psi(M, \eta, W_1, \nabla_1) - \psi(M, \eta, W_0, \nabla_0) = \frac{1}{2} \int_W p_1(W) = -12\hat{A}(W),$$

\uparrow
(Atiyah – Singer)

where $W = W_1 \cup_M W_0^{\text{opp}}$ denotes the closed spin 4-manifold obtained gluing together W_0 and W_1 along M .

Therefore the function

$$\psi(M, \eta) := \psi(M, \eta, W, \nabla) \pmod{24}$$

is well defined.

One concludes by showing that $\psi(M, \eta)$ is actually independent of the geometric string structure η , and only depending on the string cobordism class of M . Additivity is manifest from the definition, so the above integral formula defines a group homomorphism

$$\psi: \text{Bord}_3^{\text{String}} \xrightarrow{\cong} \mathbb{Z}/24\mathbb{Z}.$$

A direct computation with the canonical generator of $\text{Bord}_3^{\text{String}}$, i.e., with S^3 endowed with the trivialization of its tangent bundle coming from $S^3 \cong SU(2)$, then shows that ψ is indeed an isomorphism.

The aim of this talk is to show how the above integral formula for ψ , as well as its main properties, naturally emerge in the context of topological field theories with values in the symmetric monoidal categories associated with morphisms of abelian groups.

By $\text{Bord}_{d,d-1}^{\xi}(X)$ we will denote the symmetric monoidal category of $(d, d-1)$ -bordism with tangential structure ξ and background fields X .

The monoidal structure on $\text{Bord}_{d,d-1}^{\xi}(X)$ is given by disjoint union.

The only tangential structures we will be concerned with will be orientations, spin, and string structures; we will denote them by or , $Spin$, and $String$, respectively.

Example

Let $X = \Omega_{cl}^{d-1}$ be the smooth stack of closed $(d-1)$ -forms. Then an object of $\text{Bord}_{d,d-1}^{or}(\Omega_{cl}^{d-1})$ is given by a closed oriented $(d-1)$ -manifold M equipped with an (automatically closed) $(d-1)$ -form $\omega_{d-1;M}$. A morphism $W: M_0 \rightarrow M_1$ in $\text{Bord}_{d,d-1}^{or}(\Omega_{cl}^{d-1})$ is the datum of an oriented d -manifold W with $\partial W = M_1 \amalg M_0^{opp}$, where “opp” denotes the opposite orientation, equipped with a closed $(d-1)$ -form $\omega_{d-1;W}$ such that

$$\omega_{d-1;W}|_{M_i} = \omega_{d-1;M_i}$$

for $i = 0, 1$.

Definition

Let \mathcal{C} be a symmetric monoidal category. A $(d, d - 1)$ -dimensional \mathcal{C} -valued topological quantum field theory (TQFT for short) with tangential structure ξ and background fields X is a symmetric monoidal functor

$$Z: \text{Bord}_{d,d-1}^{\xi}(X) \rightarrow \mathcal{C}.$$

A typical target is $\mathcal{C} = \text{Vect}$, the category of vector spaces (over some fixed field \mathbb{K}). Yet there are plenty of interesting targets other than Vect . Here we will be concerned with the symmetric monoidal categories naturally associated with abelian groups and with morphisms of abelian groups.

Definition

Let $(A, +)$ be an abelian group. By A^{\otimes} we will denote the symmetric monoidal category with

$$\text{Ob}(A^{\otimes}) = A;$$

$$\text{Hom}_{A^{\otimes}}(a, b) = \begin{cases} \text{id}_a & \text{if } a = b \\ \emptyset & \text{otherwise} \end{cases}$$

The tensor product is given by the sum (or multiplication) in A and the unit object is the zero (or the unit) of A . Associators, unitors and braidings are the trivial ones.

A TQFT with tangential structure ξ and background fields X with values in A^\otimes consists into a rule that associates with any closed $(d - 1)$ -manifold M_{d-1} (with tangential structure and background fields) an element $Z(M_{d-1}) \in A$ in such a way that:

- $Z(M_{d-1} \sqcup M'_{d-1}) = Z(M_{d-1}) + Z(M'_{d-1})$ (monoidality);
- if $M_{d-1} = \partial W_d$ then $Z(M_{d-1}) = 0$ (functoriality).

Example (Stokes' theorem for closed forms)

A paradigmatic example of a TQFT with values in an abelian group is provided by Stokes' theorem. Take the stack X of background fields to be the smooth stack Ω_{cl}^{d-1} of closed $(d-1)$ -forms and let \mathbb{R}^\otimes be the symmetric monoidal category associated with the abelian group $(\mathbb{R}, +)$. Then

$$Z: \text{Bord}_{d,d-1}^{\text{or}}(\Omega_{cl}^{d-1}) \rightarrow \mathbb{R}^\otimes$$
$$(M_{d-1}, \omega_{d-1}) \mapsto \int_{M_{d-1}} \omega_{d-1}$$

is a TQFT.

Remark

Chern-Weil theory provides differential form representatives for Pontryagin classes

$$\mathbf{p}_k^{\text{CW}} : \mathbf{BSO}_{\nabla} \rightarrow \Omega_{cl}^{4k},$$

We have an induced symmetric monoidal morphism

$$\text{Bord}_{4k+1,4k}^{\text{or}}(\mathbf{BSO}_{\nabla}) \rightarrow \text{Bord}_{4k+1,4k}^{\text{or}}(\Omega_{cl}^{4k})$$

and so a TQFT

$$Z : \text{Bord}_{4k+1,4k}^{\text{or}}(\mathbf{BSO}_{\nabla}) \rightarrow \mathbb{R}^{\otimes}$$

$$(M_{4k}, P, \nabla) \mapsto \int_{M_{4k}} \mathbf{p}_k^{\text{CW}}(\nabla).$$

Remark

This TQFT descends to a TQFT with background fields \mathbf{BSO} , i.e., we have a commutative diagram

$$\begin{array}{ccc} \text{Bord}_{4k+1,4k}^{\text{or}}(\mathbf{BSO}_{\nabla}) & \longrightarrow & \mathbb{R}^{\otimes} \\ \downarrow & \nearrow & \\ \text{Bord}_{4k+1,4k}^{\text{or}}(\mathbf{BSO}) & & \end{array}$$

The tangent bundle provides a symmetric monoidal section to the forgetful morphism $\text{Bord}_{4k+1,4k}^{\text{or}}(\mathbf{BSO}) \rightarrow \text{Bord}_{4k+1,4k}^{\text{or}}$, so we get an oriented TQFT

$$\begin{aligned} Z: \text{Bord}_{4k+1,4k}^{\text{or}} &\rightarrow \mathbb{R}^{\otimes} \\ M_{4k} &\mapsto \int_{M_{4k}} \mathbf{p}_k(TM). \end{aligned}$$

Remark

The same argument applies replacing the single Pontryagin class p_k with a polynomial $\Phi = \Phi(p_1, p_2, \dots)$ in the Pontryagin classes. This way one obtains plenty of \mathbb{R} -valued oriented TQFTs. These are in particular \mathbb{R} -valued oriented cobordism invariants, and Thom's isomorphism

$$\Omega_{\bullet}^{\text{SO}} \otimes \mathbb{R} \cong \mathbb{R}[p_1, p_2, \dots]$$

implies that indeed every \mathbb{R} -valued oriented cobordism invariant is of this form.

More generally, one can associate a symmetric monoidal category with a morphism of abelian groups, as follows.

Definition

Let $\varphi_A: A_{\text{mor}} \rightarrow A_{\text{ob}}$ be a morphism of abelian groups. By φ_A^{\otimes} we will denote the symmetric monoidal category with

$$\text{Ob}(\varphi_A^{\otimes}) = A_{\text{ob}};$$

$$\text{Hom}_{\varphi_A^{\otimes}}(a, b) = \{x \in A_{\text{mor}} : a + \varphi_A(x) = b\}.$$

The composition of morphism is given by the sum in A_{mor} . The tensor product of objects and morphisms is given by the sum in A_{ob} and in A_{mor} , respectively. The unit object is the zero in A_{ob} . Associators, unitors and braidings are the trivial ones, i.e., they are given by the zero in A_{mor} .

A TQFT with values in φ_A^{\otimes} . It consists into a rule that associates with any closed $(d-1)$ -manifold M_{d-1} (with tangential structure and background fields) an element $Z(M_{d-1}) \in A_{\text{ob}}$, and with any d -manifold W_d (with tangential structure and background fields) an element $Z(W_d) \in A_{\text{mor}}$ in such a way that:

- $Z(M_{d-1} \sqcup M'_{d-1}) = Z(M_{d-1}) + Z(M'_{d-1})$ and $Z(W_d \sqcup W'_d) = Z(W_d) + Z(W'_d)$ (monoidality);
- if $M_{d-1} = \partial W_d$ then $Z(M_{d-1}) = \varphi_A(Z(W_d))$ (functoriality).

Example (Stokes' theorem)

Take as stack of background fields the smooth stack Ω^{d-1} of smooth $(d-1)$ -forms. Then we have a TQFT

$$\begin{aligned} Z: \mathit{Bord}_{d,d-1}^{\text{or}}(\Omega^{d-1}) &\rightarrow \text{id}_{\mathbb{R}}^{\otimes} \\ (M_{d-1}, \omega_{d-1}) &\mapsto \int_{M_{d-1}} \omega_{d-1} \\ (W_d, \omega_{d-1}) &\mapsto \int_{W_d} d\omega_{d-1}. \end{aligned}$$

Example (Holonomy and curvature)

A generalization of the above Example for $d = 2$ is obtained by taking $X = \mathbf{BU}(1)_{\nabla}$ and $\exp(2\pi i -)^{\otimes}$ as target category.

$$\begin{aligned} Z: \text{Bord}_{2,1}^{\text{or}}(\mathbf{BU}(1)_{\nabla}) &\rightarrow \exp(2\pi i -)^{\otimes} \\ (M_1, P, \nabla) &\mapsto \text{hol}_{M_1}(\nabla) \\ (W_2, P, \nabla) &\mapsto \frac{1}{2\pi i} \int_{W_2} F_{\nabla}, \end{aligned}$$

The fact that Z is a TQFT is encoded in the fundamental integral identity relating holonomy along the boundary and curvature in the interior:

$$\text{hol}_{\partial W_2}(\nabla) = \exp\left(\int_{W_2} F_{\nabla}\right).$$

More generally one has an $(n + 1, n)$ -dimensional TQFT as

$$\begin{aligned}
 Z: \text{Bord}_{n+1,n}^{\text{or}}(\mathbf{B}^n U(1)_{\nabla}) &\rightarrow \exp(2\pi i -)^{\otimes} \\
 (M_n, P, \nabla) &\mapsto \text{hol}_{M_n}(\nabla) \\
 (W_{n+1}, P, \nabla) &\mapsto \frac{1}{2\pi i} \int_{W_{n+1}} F_{\nabla},
 \end{aligned}$$

Example (TQFTs from spin connections)

By Brylinski–McLaughlin and F.–Schreiber–Stasheff the characteristic class $\frac{1}{2}p_1 \in H^4(BSpin; \mathbb{Z})$ refines to a commutative diagram of morphisms of smooth stacks

$$\begin{array}{ccccc}
 & & \xrightarrow{\frac{1}{2}p_1^{CW}} & & \\
 \mathbf{BSpin}_{\nabla} & \xrightarrow{\frac{1}{2}\hat{p}_1} & \mathbf{B}^3\mathbf{U}(1)_{\nabla} & \xrightarrow{\frac{1}{2\pi i}F} & \Omega_{cl}^4 \\
 \downarrow & & \downarrow & & \\
 \mathbf{BSpin} & \xrightarrow{\frac{1}{2}p_1} & \mathbf{B}^3\mathbf{U}(1), & &
 \end{array}$$

Example (TQFTs from spin connections)

This induces $(4, 3)$ -dimensional TQFT with background fields given by spin connections and target $\exp(2\pi i -)^\otimes$ given by

$$\begin{aligned} Z_{\text{Spin}} : \text{Bord}_{4,3}^{\text{or}}(\mathbf{B}\text{Spin}_\nabla) &\rightarrow \exp(2\pi i -)^\otimes \\ (M_3, P, \nabla) &\mapsto \text{hol}_{M_3} \left(\frac{1}{2} \hat{\mathbf{p}}_1(\nabla) \right) \\ (W_4, P, \nabla) &\mapsto \int_{W_4} \frac{1}{2} \mathbf{p}_1^{\text{CW}}(\nabla). \end{aligned}$$

We have a sequence of morphisms of smooth stacks

$$\mathbf{B}^3\mathbf{U}(1)_{\nabla} \rightarrow \mathbf{B}\mathbf{B}^2\mathbf{U}(1)_{\nabla} \rightarrow \mathbf{B}^2\mathbf{B}\mathbf{U}(1)_{\nabla} \rightarrow \mathbf{B}^3\mathbf{U}(1).$$

Composing this on the left with $\frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathbf{Spin}_{\nabla} \rightarrow \mathbf{B}^3\mathbf{U}(1)_{\nabla}$ we obtain maps

$$\frac{1}{2}\hat{\mathbf{p}}_1^{(i)} : \mathbf{B}\mathbf{Spin}_{\nabla} \rightarrow \mathbf{B}^{3-i}\mathbf{B}^i\mathbf{U}(1)_{\nabla},$$

for $i = 0, \dots, 3$.

Definition

For $i = 0, \dots, 3$, the smooth stack $\mathbf{BString}_{\nabla}^{(i)}$ is defined as the homotopy pullback

$$\begin{array}{ccc}
 \mathbf{BString}_{\nabla}^{(i)} & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow \\
 \mathbf{BSpin}_{\nabla} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1^{(i)}} & \mathbf{B}^{3-i}\mathbf{B}^i\mathbf{U}(1)_{\nabla}
 \end{array}$$

The stack $\mathbf{BString}_{\nabla}^{(2)}$ will be called the stack of *geometric string structures* (Waldorf).

By the pasting law for homotopy pullbacks, the defining diagram for the stack of geometric string structures can be factored as

$$\begin{array}{ccccc}
 \mathbf{BString}_{\nabla}^{(2)} & \xrightarrow{\omega_3} & \Omega^3 & \longrightarrow & * \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 \mathbf{BSpin}_{\nabla} & \xrightarrow{\frac{1}{2}\hat{p}_1} & \mathbf{B}^3\mathbf{U}(1)_{\nabla} & \longrightarrow & \mathbf{B}\mathbf{B}^2\mathbf{U}(1)_{\nabla}
 \end{array}
 \cdot$$

where both squares are homotopy pullbacks. This in particular shows that a geometric string structure comes equipped with a canonical 3-form.

From this we obtain the commutative diagram

$$\begin{array}{ccccc}
 \mathbf{BString}_{\nabla}^{(2)} & \xrightarrow{\omega_3} & \Omega^3 & & \\
 \downarrow & \lrcorner & \downarrow & \searrow d & \\
 \mathbf{BSpin}_{\nabla} & \xrightarrow{\frac{1}{2}\hat{\mathbf{p}}_1} & \mathbf{B}^3\mathbf{U}(1)_{\nabla} & \xrightarrow{\frac{1}{2\pi i}F} & \Omega_{cl}^4 \\
 & \searrow & \searrow & \nearrow & \\
 & & & \frac{1}{2}\mathbf{p}_1^{CW} &
 \end{array}$$

showing that, if $\omega_{3,M}$ is the canonical 3-form on a smooth manifold M equipped with a geometric string structure and ∇ is the underlying spin connection, then one has

$$d\omega_{3,M} = \frac{1}{2}\mathbf{p}_1^{CW}(\nabla).$$

The obstructions to lifting a topological string structure on M to a geometric string structure lie in $H^{3-i}(M; \Omega^i)$ for $i = 1, 2$. Both these groups vanish.

Proposition

Let M be a smooth manifold, and let $P: M \rightarrow \mathbf{BSpin}$ be a principal spin bundle on M . Then P can be enhanced to a geometric string structure on M if and only if $\frac{1}{2}p_1(P) = 0$.

Morphisms of abelian groups are the objects of a category whose morphisms are commutative diagrams: if $\varphi_H: H_{\text{mor}} \rightarrow H_{\text{ob}}$ and $\varphi_G: G_{\text{mor}} \rightarrow G_{\text{ob}}$ are morphisms of abelian groups, then a morphism from φ_H to φ_G is a commutative diagram of the form

$$\begin{array}{ccc} H_{\text{mor}} & \xrightarrow{\varphi_H} & H_{\text{ob}} \\ f_{\text{mor}} \downarrow & & \downarrow f_{\text{ob}} \\ G_{\text{mor}} & \xrightarrow{\varphi_G} & G_{\text{ob}} \end{array}$$

The pair $(f_{\text{ob}}, f_{\text{mor}})$ defines a symmetric monoidal functor

$$f: \varphi_H^{\otimes} \rightarrow \varphi_G^{\otimes}.$$

Definition

Given a functor $p : \mathcal{D} \rightarrow \mathcal{C}$ and an object c of \mathcal{C} , the *homotopy fiber* (or *essential fiber*) of p over c is the category $\text{hofib}(p; c)$ with objects the pairs (x, b) with x an object in \mathcal{D} and $b \in \text{hom}_{\mathcal{C}}(c, p(x))$ an isomorphism; morphisms from (x, b) to (x', b') in $\text{hofib}(p; c)$ are those morphisms $a : x \rightarrow x'$ in \mathcal{D} such that the diagram

$$\begin{array}{ccc} p(x) & \xrightarrow{p(a)} & p(x') \\ & \swarrow b \quad \searrow b' & \\ & c & \end{array}$$

commutes.

By relaxing the condition that b is an isomorphism, and allowing it to be an arbitrary morphism, we obtain the notion of *lax homotopy fiber* and denote it by $\text{hofib}_{\text{lax}}(p; c)$.

When $p : \mathcal{D} \rightarrow \mathcal{C}$ is a monoidal functor between monoidal categories, we will always take c to be the monoidal unit $\mathbf{1}_{\mathcal{C}}$ of \mathcal{C} , and simply write $\text{hofib}(p)$ and $\text{hofib}_{\text{lax}}(p)$.

The monoidal structures of \mathcal{C} and \mathcal{D} and the monoidality of p induce a natural monoidal category structure on $\text{hofib}(p)$ and on $\text{hofib}_{\text{lax}}(p)$.

The homotopy fiber of $f: \varphi_H^{\otimes} \rightarrow \varphi_G^{\otimes}$ admits a simple explicit description.

Lemma

Let

$$\begin{array}{ccc} H_{\text{mor}} & \xrightarrow{\varphi_H} & H_{\text{ob}} \\ f_{\text{mor}} \downarrow & & \downarrow f_{\text{ob}} \\ G_{\text{mor}} & \xrightarrow{\varphi_G} & G_{\text{ob}} \end{array}$$

be a commutative diagram of abelian groups, and let $f: \varphi_H^{\otimes} \rightarrow \varphi_G^{\otimes}$ be the associated monoidal functor. Then we have

$$\text{Ob}(\text{hofib}(f)) = G_{\text{mor}} \times_{G_{\text{ob}}} H_{\text{ob}}$$

$$\text{Mor}((g, h), (g', h')) = \left\{ x \in H_{\text{mor}} \text{ s.t. } \begin{cases} f_{\text{mor}}(x) = g' - g \\ \varphi_H(x) = h' - h \end{cases} \right\}.$$

Lemma

A commutative diagram of abelian groups of the form

$$\begin{array}{ccc}
 H_{\text{mor}} & \xrightarrow{\varphi_H} & H_{\text{ob}} \\
 f_{\text{mor}} \downarrow & \swarrow \lambda & \downarrow f_{\text{ob}} \\
 G_{\text{mor}} & \xrightarrow{\varphi_G} & G_{\text{ob}}
 \end{array}$$

induces a symmetric monoidal functor

$$\Xi: \text{hofib}(f) \rightarrow \ker(\varphi_G)^{\otimes}$$

acting on the objects as $(g, h) \mapsto g - \lambda(h)$.

Moreover, Ξ is an equivalence iff φ_H is an isomorphism.

We have $\mathbf{BString}_{\nabla}^{(2)} \xrightarrow{\omega_3} \Omega^3$ and so a symmetric monoidal functor

$$Z_{\text{String}} : \text{Bord}_{4,3}^{\text{or}}(\mathbf{BString}_{\nabla}^{(2)}) \rightarrow \text{Bord}_{4,3}^{\text{or}}(\Omega^3) \rightarrow \text{id}_{\mathbb{R}}^{\otimes}$$

$$(M_3, \eta) \mapsto \int_{M_3} \omega_{3;M}$$

$$(W_4, \nabla) \mapsto \int_{W_4} d\omega_{3;M} = \frac{1}{2} \int_{W_4} \mathbf{p}_1^{\text{CW}}(\nabla)$$

We also have the projection $\mathbf{BString}_{\nabla}^{(2)} \rightarrow \mathbf{BSpin}_{\nabla}$ inducing the symmetric monoidal functor

$$\mathrm{Bord}_{4,3}^{\mathrm{or}}(\mathbf{BString}_{\nabla}^{(2)}) \rightarrow \mathrm{Bord}_{4,3}^{\mathrm{or}}(\mathbf{BSpin}_{\nabla})$$

and the commutative diagram of abelian groups

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\mathrm{id}_{\mathbb{R}}} & \mathbb{R} \\ \mathrm{id}_{\mathbb{R}} \downarrow & & \downarrow \exp(2\pi i -) \\ \mathbb{R} & \xrightarrow{\exp(2\pi i -)} & \mathrm{U}(1) \end{array}$$

inducing the symmetric monoidal functor

$$(\mathrm{id}_{\mathbb{R}}, \exp(2\pi i -)): \mathrm{id}_{\mathbb{R}}^{\otimes} \rightarrow \exp(2\pi i -)^{\otimes}.$$

Lemma

The diagram of symmetric monoidal functors

$$\begin{array}{ccc}
 \text{Bord}_{4,3}^{\text{or}}(\mathbf{BString}_{\nabla}^{(2)}) & \xrightarrow{Z_{\text{String}}} & \text{id}_{\mathbb{R}}^{\otimes} \\
 \downarrow & & \downarrow (\text{id}_{\mathbb{R}}, \exp(2\pi i -)) \\
 \text{Bord}_{4,3}^{\text{or}}(\mathbf{BSpin}_{\nabla}) & \xrightarrow{Z_{\text{Spin}}} & \exp(2\pi i -)^{\otimes}
 \end{array}$$

commutes, with identity 2-cell.

We have therefore an induced monoidal functor

$$\begin{aligned}
 \text{hofib}_{\text{lax}} \left(\text{Bord}_{4,3}^{\text{or}} \left(\mathbf{BString}_{\nabla}^{(2)} \right) \rightarrow \text{Bord}_{4,3}^{\text{or}} \left(\mathbf{BSpin}_{\nabla} \right) \right) \\
 \rightarrow \text{hofib} \left(\text{id}_{\mathbb{R}}, \exp(2\pi i -) \right).
 \end{aligned}$$

$$\begin{array}{ccc}
 \mathbb{R} & \xrightarrow{\text{id}_{\mathbb{R}}} & \mathbb{R} \\
 \text{id}_{\mathbb{R}} \downarrow & \swarrow \text{id}_{\mathbb{R}} & \downarrow \exp(2\pi i -) \\
 \mathbb{R} & \xrightarrow{\exp(2\pi i -)} & U(1)
 \end{array}$$

So we have a symmetric monoidal equivalence

$$\Xi: \text{hofib}((\text{id}_{\mathbb{R}}, \exp(2\pi i -))) \rightarrow \ker(\exp(2\pi i -))^{\otimes} = \mathbb{Z}^{\otimes},$$

acting on objects as

$$\begin{aligned}
 \mathbb{R} \times_{U(1)} \mathbb{R} &\mapsto \mathbb{Z} \\
 (g, h) &\mapsto g - h
 \end{aligned}$$

Putting everything together we obtain a symmetric monoidal functor

$$Z_{\text{String}}^{\text{Spin}} : \text{hofib}_{\text{lax}} \left(\text{Bord}_{4,3}^{\text{or}}(\mathbf{BString}_{\nabla}^{(2)}) \rightarrow \text{Bord}_{4,3}^{\text{or}}(\mathbf{BSpin}_{\nabla}) \right) \rightarrow \mathbb{Z}^{\otimes}$$

Proposition

The symmetric monoidal functor $Z_{\text{String}}^{\text{Spin}}$ is the Bunke–Naumann–Redden map ψ from the Introduction.

By replacing the first fractional Pontryagin class $\frac{1}{2}p_1$ with the first Chern class c_1 one obtains an integral formula realizing the isomorphism $\text{Bord}_1^{SU} \cong \mathbb{Z}/2\mathbb{Z}$.

Question

Can one obtain an integral formula realizing the isomorphism $\text{Bord}_7^{\text{Fivebrane}} \cong \mathbb{Z}/240\mathbb{Z}$ by replacing $\frac{1}{2}p_1$ with $\frac{1}{6}p_2$?

Yes, **if** every string bundle can be endowed with a string connection. The answer to this last question is presently not clear (at least not to me).

Thanks!