Deformation classes of invertible field theories and the Freed–Hopkins conjecture

Daniel Grady

March 17 2023

Background

Background

- Want to understand the moduli stack of quantum systems with fixed symmetry type H_d.

Background

- Want to understand the moduli stack of quantum systems with fixed symmetry type H_d.
- What is a quantum system? Mathematical axioms?

Background

- Want to understand the moduli stack of quantum systems with fixed symmetry type H_d.
- What is a quantum system? Mathematical axioms?
- Idea (Freed–Hopkins): The low energy physics of a "gapped" system should be approximated by a topological field theory.

Background

- Want to understand the moduli stack of quantum systems with fixed symmetry type H_d.
- What is a quantum system? Mathematical axioms?
- Idea (Freed–Hopkins): The low energy physics of a "gapped" system should be approximated by a topological field theory. The homotopy type of the moduli stack should be determined by low energy behaviour.

Background

- Want to understand the moduli stack of quantum systems with fixed symmetry type H_d.
- What is a quantum system? Mathematical axioms?
- Idea (Freed–Hopkins): The low energy physics of a "gapped" system should be approximated by a topological field theory. The homotopy type of the moduli stack should be determined by low energy behaviour.
- By Wick-rotating to Euclidean field theories, we can use Segal's axioms to model the low energy effective theory.

Background

- Want to understand the moduli stack of quantum systems with fixed symmetry type H_d.
- What is a quantum system? Mathematical axioms?
- Idea (Freed–Hopkins): The low energy physics of a “gapped” system should be approximated by a topological field theory. The homotopy type of the moduli stack should be determined by low energy behaviour.
- By Wick-rotating to Euclidean field theories, we can use Segal's axioms to model the low energy effective theory.
- Unitarity manifests itself as reflection positivity after Wick rotation.

Theorem (Freed–Hopkins)

Let $H = \operatorname{colim}_{d\to\infty} H_d$ be a stable symmetry type. There is a bijective correspondence

$$\left\{\begin{array}{l}\text{deformation classes of reflection}\\ \text{positive invertible } d\text{-dimensional}\\ \text{extended topological field theories}\\ \text{with symmetry type } (H_d, \rho_d)\end{array}\right\} \cong [MTH, \Sigma^{d+1} I_{\mathbb{Z}(1)}]_{\mathrm{tor}}$$

Theorem (Freed–Hopkins)

Let $H = \operatorname{colim}_{d\to\infty} H_d$ be a stable symmetry type. There is a bijective correspondence

$$\left\{\begin{array}{l}\textit{deformation classes of reflection} \\ \textit{positive invertible } d\textit{-dimensional} \\ \textit{extended topological field theories} \\ \textit{with symmetry type } (H_d, \rho_d)\end{array}\right\} \cong [MTH, \Sigma^{d+1} I_{\mathbb{Z}(1)}]_{\mathrm{tor}}$$

- Where MTH is the Thom spectrum associated to the stable symmetry type.

Theorem (Freed–Hopkins)

Let $H = \mathrm{colim}_{d\to\infty} H_d$ be a stable symmetry type. There is a bijective correspondence

$$\left\{\begin{array}{l}\textit{deformation classes of reflection}\\ \textit{positive invertible } d\textit{-dimensional}\\ \textit{extended topological field theories}\\ \textit{with symmetry type } (H_d, \rho_d)\end{array}\right\} \cong [MTH, \Sigma^{d+1} I_{\mathbb{Z}(1)}]_{\mathrm{tor}}$$

- Where MTH is the Thom spectrum associated to the stable symmetry type.
- $\Sigma^{d+1} I_{\mathbb{Z}(1)}$ is the $(d+1)$-fold suspension of the Anderson dual of the sphere.

Conjecture (Freed–Hopkins)

Let $H = \operatorname{colim}_{d\to\infty} H_d$ be a stable symmetry type. There is a bijective correspondence

$$\left\{\begin{array}{l}\text{deformation classes of reflection} \\ \text{positive invertible } d\text{-dimensional} \\ \text{extended field theories with sym-} \\ \text{metry type } (H_d, \rho_d)\end{array}\right\} \cong [MTH, \Sigma^{d+1} I_{\mathbb{Z}(1)}]$$

- Where MTH is the Thom spectrum associated to the stable symmetry type.
- $\Sigma^{d+1} I_{\mathbb{Z}(1)}$ is the $(d+1)$-fold suspension of the Anderson dual of the sphere.

Toy example

Toy example

- Consider a 1-d sigma model with target manifold X:

Toy example

- Consider a 1-d sigma model with target manifold X:

$$(x : * \to X) \mapsto V_x$$
$$(\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y) \mapsto (L_\gamma : V_x \to V_y)$$

- If we require that the field theory is smooth, we get the data of a vector bundle with connection on X.

Toy example

- Consider a 1-d sigma model with target manifold X:

$$(x : * \to X) \mapsto V_x$$
$$(\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y) \mapsto (L_\gamma : V_x \to V_y)$$

- If we require that the field theory is smooth, we get the data of a vector bundle with connection on X.
- Every vector bundle with connection on X also gives rise to a field theory.

Toy example

- Consider a 1-d sigma model with target manifold X:

$$(x : * \to X) \mapsto V_x$$
$$(\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y) \mapsto (L_\gamma : V_x \to V_y)$$

- If we require that the field theory is smooth, we get the data of a vector bundle with connection on X.
- Every vector bundle with connection on X also gives rise to a field theory. In fact, we have an equivalnece at the level of moduli stacks (Berwick-Evans, Pavlov):

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^X, \mathrm{Vect}) \cong \mathrm{Vect}_\nabla^\times(X)$$

Toy example

- Consider a 1-d sigma model with target manifold X:

$$(x : * \to X) \mapsto V_x$$
$$(\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y) \mapsto (L_\gamma : V_x \to V_y)$$

- If we require that the field theory is smooth, we get the data of a vector bundle with connection on X.
- Every vector bundle with connection on X also gives rise to a field theory. In fact, we have an equivalnece at the level of moduli stacks (Berwick-Evans, Pavlov):

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^X, \mathrm{Vect}) \cong \mathrm{Vect}_{\nabla}^{\times}(X)$$

- If L_γ only depends on the homotopy class of γ, the theory is topological (homotopy invariant).

Toy example

- Consider a 1-d sigma model with target manifold X:

$$(x : * \to X) \mapsto V_x$$
$$(\gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y) \mapsto (L_\gamma : V_x \to V_y)$$

- If we require that the field theory is smooth, we get the data of a vector bundle with connection on X.
- Every vector bundle with connection on X also gives rise to a field theory. In fact, we have an equivalnece at the level of moduli stacks (Berwick-Evans, Pavlov):

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^X, \mathrm{Vect}) \cong \mathrm{Vect}_{\nabla}^{\times}(X)$$

- If L_γ only depends on the homotopy class of γ, the theory is topological (homotopy invariant). The corresponding vector bundle is flat.

- In the case of invertible field theories, the FT take values in $\mathrm{Line} \subset \mathrm{Vect}$.

- In the case of invertible field theories, the FT take values in $\mathrm{Line} \subset \mathrm{Vect}$.
- The inclusion of isomorphism classes of topological field theories into all field theories is

$$\begin{array}{ccc} [X, \mathbf{B}(\mathbb{C}^\times)^\delta] = H^1(X;\mathbb{C}^\times) & \longrightarrow & [X, \mathbf{B}_\nabla \mathbb{C}^\times] \\ & \searrow^{\beta} & \downarrow \text{def. classes} \\ & & H^2(X;\mathbb{Z}) \end{array}$$

- The image of β is the torsion subgroup.

Reflection positivity

- Fix a symmetry type H_d (compact Lie group) and a representation $\rho_d : H_d \to O_d$.

Reflection positivity

- Fix a symmetry type H_d (compact Lie group) and a representation $\rho_d : H_d \to O_d$. We require that $SO_d \subset \rho_d(H_d)$.

Reflection positivity

- Fix a symmetry type H_d (compact Lie group) and a representation $\rho_d : H_d \to O_d$. We require that $SO_d \subset \rho_d(H_d)$.
- If $d > 3$ and $\rho_d(H_d) = SO_d$, then

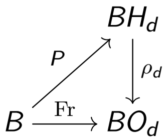
$$H_d = K \times \mathrm{Spin}_d / \langle (k_0, -1) \rangle \quad K = \ker(\rho_d)$$

Reflection positivity

- Fix a symmetry type H_d (compact Lie group) and a representation $\rho_d : H_d \to O_d$. We require that $SO_d \subset \rho_d(H_d)$.
- If $d > 3$ and $\rho_d(H_d) = SO_d$, then

$$H_d = K \times \mathrm{Spin}_d / \langle (k_0, -1) \rangle \quad K = \ker(\rho_d)$$

- An (H_d, ρ_d)-structure on a bordism an H_d-subbundle $P \to B$ of the bundle of orthonormal frames. Equivalently, it is a lift:



- A choice of hyperplane reflection $\sigma \in O_d$ gives rise to an automorphism ϕ_σ of H_d.

- A choice of hyperplane reflection $\sigma \in O_d$ gives rise to an automorphism ϕ_σ of H_d.
- ϕ_σ induces an involution β on the bordism category, sending $(B, P \to B)$ to $(B, P' \to B)$. The bundle $P' \to B$ is the H_d-bundle obtained by precomposing the original H_d-action on P by the automorphism ϕ_σ.

- A choice of hyperplane reflection $\sigma \in O_d$ gives rise to an automorphism ϕ_σ of H_d.
- ϕ_σ induces an involution β on the bordism category, sending $(B, P \to B)$ to $(B, P' \to B)$. The bundle $P' \to B$ is the H_d-bundle obtained by precomposing the original H_d-action on P by the automorphism ϕ_σ.
- A *reflection structure* on a field theory

$$Z : \mathrm{Bord}_d^{H_d} \to \mathrm{Vect}$$

is a natural iso

$$Z(\beta B) \cong \overline{Z(B)}.$$

- A choice of hyperplane reflection $\sigma \in O_d$ gives rise to an automorphism ϕ_σ of H_d.
- ϕ_σ induces an involution β on the bordism category, sending $(B, P \to B)$ to $(B, P' \to B)$. The bundle $P' \to B$ is the H_d-bundle obtained by precomposing the original H_d-action on P by the automorphism ϕ_σ.
- A *reflection structure* on a field theory

$$Z : \mathrm{Bord}_d^{H_d} \to \mathrm{Vect}$$

  is a natural iso

$$Z(\beta B) \cong \overline{Z(B)}.$$

- Given a field theory with reflection structure Z, we have a hermitian form

$$h : Z(B) \otimes \overline{Z(B)} \cong Z(B) \otimes Z(\beta B) \cong Z(B) \otimes Z(B^\vee) \to \mathbb{C}.$$

- A choice of hyperplane reflection $\sigma \in O_d$ gives rise to an automorphism ϕ_σ of H_d.
- ϕ_σ induces an involution β on the bordism category, sending $(B, P \to B)$ to $(B, P' \to B)$. The bundle $P' \to B$ is the H_d-bundle obtained by precomposing the original H_d-action on P by the automorphism ϕ_σ.
- A *reflection structure* on a field theory

$$Z : \mathrm{Bord}_d^{H_d} \to \mathrm{Vect}$$

is a natural iso

$$Z(\beta B) \cong \overline{Z(B)}.$$

- Given a field theory with reflection structure Z, we have a hermitian form

$$h : Z(B) \otimes \overline{Z(B)} \cong Z(B) \otimes Z(\beta B) \cong Z(B) \otimes Z(B^\vee) \to \mathbb{C}.$$

- If h is positive definite, we say that the field theory is *positive*.

Full-extended reflection positive invertible theories

Full-extended reflection positive invertible theories

- Restricting to invertibe field theories, we replace Vect by Line. Going to fully-extended invertible field theories, Freed–Hopkins replace Line by $\Sigma^{d+1} I_{\mathbb{Z}(1)}$.

Full-extended reflection positive invertible theories

- Restricting to invertibe field theories, we replace Vect by Line. Going to fully-extended invertible field theories, Freed–Hopkins replace Line by $\Sigma^{d+1} I_{\mathbb{Z}(1)}$.

Theorem (GMTW, Schommer-Pries)

The homotopy type of the fully-extended bordism category $\mathrm{Bord}_d^{H_d}$ is $\Sigma^d MTH_d$.

- A field theory $Z : \mathrm{Bord}_d^{H_d} \to \Sigma^{d+1} I_{\mathbb{Z}(1)}$ canonically factors as

$$\begin{array}{ccc} \mathrm{Bord}_d^{H_d} & \longrightarrow & \Sigma^{d+1} I_{\mathbb{Z}(1)} \\ \downarrow & \nearrow & \\ \Sigma^d MTH_d & & \end{array}$$

- The involution β induces an involution on $\Sigma^d MTH_d$.

- By definition, the Anderson dual $I_{\mathbb{Z}(1)}$ sits in a long homotopy fiber/cofiber sequence

$$\ldots \to \Sigma^d I_{\mathbb{Z}(1)} \to \Sigma^d I_{\mathbb{C}} \to \Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)} \to \ldots.$$

- Complex conjugation on $\mathbb{C}$ induces a $\mathbb{Z}/2$-action on $I_{\mathbb{C}} \cong H\mathbb{C}$.

- By definition, the Anderson dual $I_{\mathbb{Z}(1)}$ sits in a long homotopy fiber/cofiber sequence

$$\dots \to \Sigma^d I_{\mathbb{Z}(1)} \to \Sigma^d I_{\mathbb{C}} \to \Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)} \to \dots .$$

- Complex conjugation on $\mathbb{C}$ induces a $\mathbb{Z}/2$-action on $I_{\mathbb{C}} \cong H\mathbb{C}$.
- The space of $\mathbb{Z}/2$-actions on $I_{\mathbb{Z}(1)}$ that is compatible with the conjugation action is not contractible.

- By definition, the Anderson dual $I_{\mathbb{Z}(1)}$ sits in a long homotopy fiber/cofiber sequence

$$\ldots \to \Sigma^d I_{\mathbb{Z}(1)} \to \Sigma^d I_{\mathbb{C}} \to \Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)} \to \ldots.$$

- Complex conjugation on $\mathbb{C}$ induces a $\mathbb{Z}/2$-action on $I_{\mathbb{C}} \cong H\mathbb{C}$.
- The space of $\mathbb{Z}/2$-actions on $I_{\mathbb{Z}(1)}$ that is compatible with the conjugation action is not contractible. Freed and Hopkins make a prefered choice of action γ.
- The deformation classes of (nontopological) reflection theories are conjectured correspond to maps of equivariant spectra

$$Z : (\Sigma^d MTH_d)^\beta \to (\Sigma^{d+1} I_{\mathbb{Z}(1)})^\gamma.$$

- I will not discuss positivity in the extended setting.

- The map $\Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)}$ in the long fiber/cofiber sequence induces a map

$$[\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}] \to [\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)}]$$

whose image is the torsion subgroup, i.e., the deformation classes of topological invertible field theories.

- The map $\Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)}$ in the long fiber/cofiber sequence induces a map

$$[\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}] \to [\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)}]$$

  whose image is the torsion subgroup, i.e., the deformation classes of topological invertible field theories.
- Same equivariantly.

- The map $\Sigma^d I_{\mathbb{C}^\times} \to \Sigma^{d+1} I_{\mathbb{Z}(1)}$ in the long fiber/cofiber sequence induces a map

$$[\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}] \to [\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)}]$$

  whose image is the torsion subgroup, i.e., the deformation classes of topological invertible field theories.
- Same equivariantly.
- To prove the theorem, one must do the following:
  1. Enhance the H_d-structure on bordisms to a *differential* H_d-structure (including connections on bundles).
  2. Construct a geometric refinement $\Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}$ whose "deformation spectrum" is $\Sigma^{d+1} I_{\mathbb{Z}(1)}$. The refinement should see the smooth structure of $\mathbb{C}^\times$.

The main theorem

Theorem (G.)

The following spaces are isomorphic in the homotopy category of spaces

1. *Smooth deformations of field theories with smooth* (H_d, ρ_d)*-structure:* $I_d(\mathcal{H}_d) := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$
2. *Smooth deformations of field theories with differential* (H_d, ρ_d)*-structure:* $I_d(\mathcal{H}_d^\nabla) := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^\nabla}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$
3. *Smooth deformations of field theories with flat* (H_d, ρ_d)*-structure:* $I_d(\mathcal{H}_d^{\mathrm{fl}}) := \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\mathrm{fl}}}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$
4. *The space of morphisms of spectra:* $\mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$.

- In our formalism, geometric structures on bordisms are encoded by simplicial presheaves on FEmb_d.

- In our formalism, geometric structures on bordisms are encoded by simplicial presheaves on FEmb_d.
- Objects of FEmb_d are submersions $M \to U$ with d-dimensional fibers, $U \subset \mathbb{R}^n$, $U \cong \mathbb{R}^n$.

- In our formalism, geometric structures on bordisms are encoded by simplicial presheaves on FEmb_d.
- Objects of FEmb_d are submersions $M \to U$ with d-dimensional fibers, $U \subset \mathbb{R}^n$, $U \cong \mathbb{R}^n$. Morphisms are fiberwise open embeddings.

- In our formalism, geometric structures on bordisms are encoded by simplicial presheaves on FEmb_d.
- Objects of FEmb_d are submersions $M \to U$ with d-dimensional fibers, $U \subset \mathbb{R}^n$, $U \cong \mathbb{R}^n$. Morphisms are fiberwise open embeddings.
- The presheaf $\mathcal{H}_d^{\nabla}$ is given by the homotopy pullback

$$\begin{array}{ccc} \mathcal{H}_d^{\nabla} & \longrightarrow & \mathbf{B}_{\nabla} H_d \\ \downarrow & & \downarrow \\ \mathrm{Riem} & \xrightarrow{\mathrm{LC}} & \mathbf{B}_{\nabla}\mathrm{GL}_d \end{array}$$

- In our formalism, geometric structures on bordisms are encoded by simplicial presheaves on FEmb_d.
- Objects of FEmb_d are submersions $M \to U$ with d-dimensional fibers, $U \subset \mathbb{R}^n$, $U \cong \mathbb{R}^n$. Morphisms are fiberwise open embeddings.
- The presheaf $\mathcal{H}_d^{\nabla}$ is given by the homotopy pullback

$$\begin{array}{ccc} \mathcal{H}_d^{\nabla} & \longrightarrow & \mathbf{B}_{\nabla} H_d \\ \downarrow & & \downarrow \\ \mathrm{Riem} & \xrightarrow{\mathrm{LC}} & \mathbf{B}_{\nabla} \mathrm{GL}_d \end{array}$$

- A vertex in $\mathcal{H}_d^{\nabla}(M \to U)$ is a fiberwise principal H_d-bundle with connection, a Riemannian metric and a connection preserving isomorphism of bundles between the associated bundle with connection with the fiberwise tangent bundle, with the Levi-Civita connection.

- A choice of hyperplane reflection gives an automorphism of $\mathbf{B}_{\nabla} H_d$, which induces an automorphism of $\mathcal{H}_d^{\nabla}$.

- A choice of hyperplane reflection gives an automorphism of $\mathbf{B}_\nabla H_d$, which induces an automorphism of $\mathcal{H}_d^\nabla$.
- We again have an involution at the level of bordism categories

$$\beta : \mathrm{Bord}_d^{\mathcal{H}_d^\nabla} \to \mathrm{Bord}_d^{\mathcal{H}_d^\nabla}$$

- The object $I_{\mathbb{C}^\times_{\mathrm{sm}}}$ is a sheaf of spectra on cartesian spaces, defined using Brown representability (Patchkoria and Pstragowski).

- A choice of hyperplane reflection gives an automorphism of $\mathbf{B}_\nabla H_d$, which induces an automorphism of $\mathcal{H}_d^\nabla$.
- We again have an involution at the level of bordism categories

$$\beta : \mathrm{Bord}_d^{\mathcal{H}_d^\nabla} \to \mathrm{Bord}_d^{\mathcal{H}_d^\nabla}$$

- The object $I_{\mathbb{C}^\times_{\mathrm{sm}}}$ is a sheaf of spectra on cartesian spaces, defined using Brown representability (Patchkoria and Pstragowski).
- That is, $I_{\mathbb{C}^\times_{\mathrm{sm}}}$ satisfies:

$$[X, I_{\mathbb{C}^\times_{\mathrm{sm}}}] \cong \mathsf{hom}(\tilde{\pi}_0(X), \mathbb{C}^\times),$$

where the hom is taken in sheaves of abelian groups.

- A choice of hyperplane reflection gives an automorphism of $\mathbf{B}_\nabla H_d$, which induces an automorphism of $\mathcal{H}_d^\nabla$.
- We again have an involution at the level of bordism categories

$$\beta : \mathrm{Bord}_d^{\mathcal{H}_d^\nabla} \to \mathrm{Bord}_d^{\mathcal{H}_d^\nabla}$$

- The object $I_{\mathbb{C}^\times_{\mathrm{sm}}}$ is a sheaf of spectra on cartesian spaces, defined using Brown representability (Patchkoria and Pstragowski).
- That is, $I_{\mathbb{C}^\times_{\mathrm{sm}}}$ satisfies:

$$[X, I_{\mathbb{C}^\times_{\mathrm{sm}}}] \cong \mathrm{hom}(\tilde{\pi}_0(X), \mathbb{C}^\times),$$

  where the hom is taken in sheaves of abelian groups.
- There is a $\mathbb{Z}/2$-action on $I_{\mathbb{C}^\times_{\mathrm{sm}}}$, compatible with complex conjugation.

- Recall the functor

$$\int : Sh_{\infty}(Cart; Sp) \to Sp,$$

- Recall the functor

$$\int : Sh_\infty(Cart; Sp) \to Sp,$$

which is homotopy left adjoint to the locally constant functor.

- The functor $I_{\mathbb{C}^\times}$ sits in a long fiber cofiber sequence

$$\ldots \to \Sigma^d I_{\mathbb{Z}(1)} \to \Sigma^d I_{\mathbb{C}_{\mathrm{sm}}} \to \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}} \to \Sigma^{d+1} I_{\mathbb{Z}(1)} \to \ldots.$$

- Recall the functor

$$\int : Sh_\infty(Cart; Sp) \to Sp,$$

which is homotopy left adjoint to the locally constant functor.
- The functor $I_{\mathbb{C}^\times}$ sits in a long fiber cofiber sequence

$$\ldots \to \Sigma^d I_{\mathbb{Z}(1)} \to \Sigma^d I_{\mathbb{C}_{\mathrm{sm}}} \to \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}} \to \Sigma^{d+1} I_{\mathbb{Z}(1)} \to \ldots.$$

- Since $\mathbb{C}$ deformation retracts through group homomorphisms to 0, applying $\int$ gives an equivalence

$$\int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}} \xrightarrow{\simeq} \Sigma^{d+1} I_{\mathbb{Z}(1)}$$

- The homotopy type of the bordism category $\mathrm{Bord}_d^{\mathcal{H}_d}$ is computed as the composition of two functors

$$C^\infty \mathrm{Cat}_{\infty,d}^{\otimes} \xrightarrow{|\cdot|} Sh(Cart; Sp) \xrightarrow{\int} Sp$$

Theorem (G., Pavlov)

Fix $d \geq 0$. We have an equivalence

$$\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\mathrm{fl}}}| \simeq \Sigma^d MTH_d$$

- The canonical map $\mathcal{H}_d^{\mathrm{fl}} \to \mathcal{H}_d^{\nabla}$ of flat bundles into all bundles induces an equivalence

$$\int \mathrm{Bord}_d^{\mathcal{H}_d^{\mathrm{fl}}} \to \int \mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}$$

- Argument has an h-principle flavor.

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \overset{\simeq}{\leftarrow} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}})$$

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \overset{\simeq}{\leftarrow} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}})$$

and a map (taking deformations)

$$\mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \xrightarrow{\int} \mathrm{Fun}^{\otimes}(\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}).$$

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \xleftarrow{\simeq} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}})$$

and a map (taking deformations)

$$\mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \xrightarrow{\int} \mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}).$$

The right side was computed as

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \overset{\simeq}{\leftarrow} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$$

  and a map (taking deformations)

$$\mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \overset{\int}{\to} \mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}).$$

  The right side was computed as

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Since $\Sigma^d I_{\mathbb{C}^\times}$ is homotopy invariant, we have also have

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times})$$

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \overset{\simeq}{\leftarrow} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}})$$

and a map (taking deformations)

$$\mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \overset{\int}{\to} \mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}).$$

The right side was computed as

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^\times_{\mathrm{sm}}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Since $\Sigma^d I_{\mathbb{C}^\times}$ is homotopy invariant, we have also have

$$\mathrm{Fun}^{\otimes}(\textstyle\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^\times}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times})$$

- The canonical inclusion $I_{\mathbb{C}^\times} \to I_{\mathbb{C}^\times_{\mathrm{sm}}}$ therefore induces a map

$$\mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^\times}) \to \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \xleftarrow{\simeq} \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}})$$

and a map (taking deformations)

$$\mathrm{Fun}^{\otimes}(|\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \xrightarrow{\int} \mathrm{Fun}^{\otimes}(\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}).$$

The right side was computed as

$$\mathrm{Fun}^{\otimes}(\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \int \Sigma^d I_{\mathbb{C}^{\times}_{\mathrm{sm}}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Since $\Sigma^d I_{\mathbb{C}^{\times}}$ is homotopy invariant, we have also have

$$\mathrm{Fun}^{\otimes}(\int |\mathrm{Bord}_d^{\mathcal{H}_d^{\nabla}}|, \Sigma^d I_{\mathbb{C}^{\times}}) \simeq \mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^{\times}})$$

- The canonical inclusion $I_{\mathbb{C}^{\times}} \to I_{\mathbb{C}^{\times}_{\mathrm{sm}}}$ therefore induces a map

$$\mathrm{Map}(\Sigma^d MTH_d, \Sigma^d I_{\mathbb{C}^{\times}}) \to \mathrm{Map}(\Sigma^d MTH_d, \Sigma^{d+1} I_{\mathbb{Z}(1)})$$

- Taking π_0 the image is the torsion subgroup of deformation classes of topological theories.

Thank you!