T-duality for loop spaces, or equivalently for the 1D sigma model

M-Theory and Mathematics: classical and quantum aspects

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Collaborators and references

[HM]

Fei Han and V. M.,
Exotic twisted equivariant cohomology of loop spaces, twqisted Bismut-Chern character and T-duality.
Communications in Mathematical Physics,
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[BEM]

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Contents

Motivation for some constructions on loop space

Atiyah, working out an idea of Witten, revealed the remarkable fact that the **index of the Dirac operator** acting on the spin complex of a compact spin manifold can be formally interpreted as an **integral of an equivariantly closed differential form over loop space** (wrt the rotation circle action on loops).

It is only formal because integration over infinite dimensional loop space is not well defined.

Then a formal application of the **localisation formula of Duistermaat-Heckman** leads to the index theorem of Atiyah-Singer for the Dirac operator.

Motivation for some constructions on loop space

Bismut extended this approach to a **Dirac operator twisted by a vector bundle with connection**. In doing so, for a vector bundle with connection, he constructed an **equivariantly closed form on the loop space**, lifting to loop space the Chern character form of the vector bundle with connection.

Bismut's construction of the **loop space refinement of the Chern character form** can be viewed roughly as the trace of the solution to an equation analogous to the holonomy of a connection, and is expressed as the trace of a **path ordered exponential**.

Motivation for some constructions on loop space

It was shown by **Jones-Petrack** that a completed version of equivariant cohomology of loopspace LZ with respect to the rotation circle action, localises to the ordinary cohomology of Z, that is, establishing a statement of Witten

$$h^\bullet_{\mathbb{T}}(LZ) \cong H^\bullet(Z)[u, u^{-1}]]$$

Bismut refined the Chern character in such a way that the following diagram commutes,

$$\begin{array}{ccc} K^\bullet(Z) & \xrightarrow{\;BCh\;} & h^\bullet_{\mathbb{T}}(LZ) \\ & {\scriptstyle Ch}\searrow \quad \swarrow{\scriptstyle res} & \\ & H^\bullet(Z)[u, u^{-1}]] & \end{array}$$

where *res* is the localisation map,

Motivation for some constructions on loop space

This talk is concerned with the analog of this result is for **twisted cohomology**, $H^\bullet(Z, H)$ where H is a closed degree 3 form on Z with integral periods, i.e. $[H] \in H^3(Z; \mathbb{Z})$.
Here $H^\bullet(Z, H) = H^\bullet(\Omega^{odd/even}(Z), d + H\wedge)$ is a $\mathbb{Z}_2$-graded cohomology theory, coinciding with $H^\bullet(Z)$ when $H = 0$.

It was first studied by Rohm-Witten (1986), and arose in String Theory as the **charge group** classifying D-brane charges at least rationally. However, it has many applications in mathematics such as twisted eta invariants, twisted analytic torsion, etc.

Motivation for some constructions on loop space

Fei Han and I defined an **exotic equivariant cohomology** - a key innovation is the construction of a canonical ***S*^{1}-flat superconnection** on the the holonomy line bundle of a gerbe with connection, satisfying the **localisation formula**

$$h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \cong H^\bullet(Z, H)[u, u^{-1}]]$$

We also defined a **loop space refinement** of the twisted Chern character of Bouwknegt-Carey-VM-Murray-Stevenson in such a way that the following diagram commutes,

$$\begin{array}{ccc} K^\bullet(Z, H) & \xrightarrow{\quad BCh_H \quad} & h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \\ & {\scriptstyle Ch_H} \searrow \qquad \swarrow {\scriptstyle res} & \\ & H^\bullet(Z, H)[u, u^{-1}]] & \end{array}$$

where *res* is the localisation map.

T-duality from a loop space perspective

T-duality done on spacetime, is but a shadow of T-duality on **loop space** of spacetime, which is the configuration space of (closed) strings in String Theory.

Therefore we propose that the charge group of the RR-fields is this new cohomology theory on loop space, and moreover that the RR fields themselves are differential forms on loop space with coefficients in the holonomy line bundle that are in the nullspace of the S^1-flat superconnection.

Gerbes

Consider a pair (Z, H), where Z is a spacetime and H is a background flux, i.e. a closed 3-form on Z with $\mathbb{Z}$ periods.

We want to study open covers $\{U_\alpha\}$ of Z such that the space of loops $\{LU_\alpha\}$ is an open cover of $LZ = C^\infty(S^1, Z)$.
The usual Cech open cover of Z consisting of a convex open cover of Z does **not** satisfy this property.
Suppose that $\{U_\alpha\}$ is a maximal open cover of Z with the property that $H^i(U_{\alpha_I}) = 0$ for $i = 2, 3$ where $U_{\alpha_I} = \bigcap_{i \in I} U_{\alpha_i}$, $|I| < \infty$. Such an open cover is a **Brylinski open cover** of Z.
It is easy to see that $\{LU_\alpha\}$ is an open cover of LZ.

Let H a closed 3-form on Z with integral periods. Then $H\big|_{U_\alpha} = dB_\alpha$ since $H^3(U_\alpha) = 0$ where $B_\alpha \in \Omega^2(U_\alpha)$. Also $B_\beta - B_\alpha = dA_{\alpha\beta}$ since $H^2(U_\alpha \cap U_\beta) = 0$. Then (H, B, A) defines a connective structure (or connection) for a **gerbe** $\mathcal{G}_B$ on Z.

Gerbes

More precisely, a **gerbe** $\mathcal{G}$ on Z is a collection of line bundles $\{L_{\alpha\beta}\}$ on double overlaps, $L_{\alpha\beta} \to U_{\alpha\beta} = U_\alpha \cap U_\beta$ such that on triple overlaps $U_{\alpha\beta\gamma}$ there is a trivialization

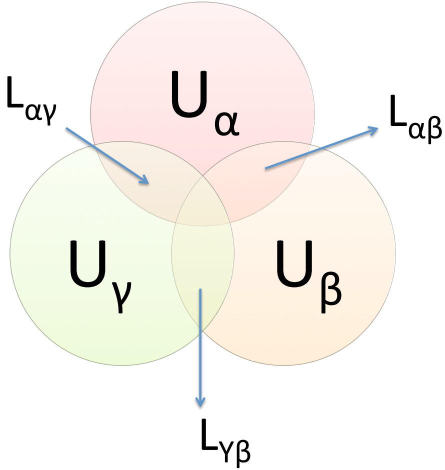
$$\phi_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} \xrightarrow{\cong} \mathbb{C}$$

Then $\{\phi_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Cech 2-cocycle representing the **Dixmier-Douady invariant** of the gerbe in $H^3(Z, \mathbb{Z})$.
Upto equivalence, gerbes on Z are classified by $H^3(Z, \mathbb{Z})$.

A **trivial gerbe** $\{L_{\alpha\beta}\}$ is of the form $L_{\alpha\beta} = L_\alpha \otimes L^*_\beta$, where $\{L_\alpha \to U_\alpha\}$ is a collection of line bundles.

Gerbes

GERBE



Example: Spin$^{\mathbb{C}}$-gerbes

Let $\{g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)\}$ denote the set of transition functions for the oriented orthonormal frame bundle of Z,

$$U(1) \to Spin^{\mathbb{C}}(n) \to SO(n)$$

is the defining nontrivial central extension. Let $L \to SO(n)$ be the associated line bundle, $L = Spin^{\mathbb{C}}(n) \times_{U(1)} \mathbb{C}$. Then the gerbe $\{L_{\alpha\beta} = g^*_{\alpha\beta}(L)\}$ is called the **Spin$^{\mathbb{C}}$-gerbe** of Z. The Dixmier-Douady class of this gerbe is equal to $W_3(Z)$, the 3rd integral Stiefel-Whitney class of Z. ***So every oriented manifold has a Spin$^{\mathbb{C}}$-gerbe.***

This construction also works for the oriented orthonormal frame bundle of any oriented vector bundle E over Z.

Example: *PU*-gerbes

Let $\{g_{\alpha\beta} : U_{\alpha\beta} \to PU\}$ denote the set of transition functions for a principal *PU*-bundle P over Z,

$$U(1) \to U \to PU$$

is the defining nontrivial central extension.

Let $L \to PU$ be the associated line bundle, $L = U \times_{U(1)} \mathbb{C}$. Then the gerbe $\{L_{\alpha\beta} = g^*_{\alpha\beta}(L)\}$ is called the *PU*-gerbe of P over Z.

The Dixmier-Douady class of this gerbe is equal to $DD(P)$.

Gerbes, connections and their holonomy line bundle

A **connection** on the gerbe $\mathcal{G}_B$ is $\{(L_{\alpha\beta}, \nabla^L_{\alpha\beta})\}$, a collection of line bundles $L_{\alpha\beta} \to U_{\alpha\beta}$ such that there is an isomorphism $L_{\alpha\beta} \otimes L_{\beta\gamma} \cong L_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$ and collection of connections $\{\nabla^L_{\alpha\beta}\}$ such that $\nabla^L_{\alpha\beta} = d + A_{\alpha\beta}$ (note that as $H^2(U_\alpha \cap U_\beta) = 0$, the bundle $L_{\alpha\beta}$ is trivial). Then we have $(\nabla^L_{\alpha\beta})^2 = F^L_{\alpha\beta} = B_\beta - B_\alpha$.

The **holonomy** of this gerbe is a line bundle $\mathcal{L}^B \to LZ$ over the loop space LZ. $\mathcal{L}^B$ has $\mathbb{T}$-invariant Brylinski local sections $\{\sigma_\alpha\}$ with respect to $\{LU_\alpha\}$ such that the transition functions are $\{e^{-\sqrt{-1}\tau(A_{\alpha_\beta})}\}$, i.e. $\sigma_\alpha = e^{-\sqrt{-1}\tau(A_{\alpha_\beta})}\sigma_\beta$,
$\tau : \Omega^\bullet(U_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LU_{\alpha_I})$ is the transgression map defined as $\tau(\xi_I) = \int_{\mathbb{T}} ev^*(\xi_I)$, $\xi_I \in \Omega^\bullet(U_{\alpha_I})$. Here ev is the evaluation map $ev : \mathbb{T} \times LU_{\alpha_I} \to U_{\alpha_I} : (t, \gamma) \to \gamma(t)$.

Gerbes and their holonomy line bundle

The holonomy line bundle $\mathcal{L}^B$ on loopspace LZ comes with a natural connection, whose definition with respect to the basis $\{\sigma_\alpha\}$ is $\nabla^{\mathcal{L}^B} = d - \sqrt{-1}\tau(B_\alpha)$. The curvature of the connection $\nabla^{\mathcal{L}^B}$ is $F_B = (\nabla^{\mathcal{L}^B})^2 = -\tau(H)$ is the transgression of the minus 3-curvature H of the gerbe $\mathcal{G}_B$.

Observe that $\mathcal{L}^B$ is never flat if $H \neq 0$.

Consider $\Omega^\bullet(LZ, \mathcal{L}^B) =$ the space of differential forms on loop space LZ with values in the holonomy line bundle $\mathcal{L}^B \to LZ$ of the gerbe $\mathcal{G}_B$ on Z.

Induced tensors on loop space

Let $\omega \in \Omega^i(Z)$. Define $\hat{\omega}_s \in \Omega^i(LZ)$ for $s \in [0,1]$ by

$$\hat{\omega}_s(X_1, \ldots, X_i)(\gamma) = \omega(X_1\big|_{\gamma(s)}, \ldots, X_i\big|_{\gamma(s)})$$

for $\gamma \in LZ$ and $X_1, \ldots, X_i$ are vector fields on LZ defined near γ. Then one checks that $d\hat{\omega}_s = \widehat{d\omega}_s$. The i-form

$$\bar{\omega} = \int_0^1 \hat{\omega}_s ds \quad \in \Omega^i(LZ)$$

is $\mathbb{T}$-invariant, that is, $L_K(\bar{\omega}) = 0$ and $d\bar{\omega} = \overline{d\omega}$.
Moreover $\tau(\omega) = i_K\bar{\omega}$ and that $\bar{\omega}$ restricts to ω on the submanifold of constant loops.

Exotic twisted equivariant cohomology of loop space

Let H be as before and $\bar{H} \in \Omega^3(LZ)$ be the associated closed 3-form on LZ. Define $D_{\bar{H}} = \nabla^{\mathcal{L}^B} - i_K + \bar{H}$. Then we compute,

Lemma

$(D_{\bar{H}})^2 = 0$ *on* $\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}$.

Proof.

Let $\{U_\alpha\}$ be a Brylinski open cover of Z. Then $\bar{H}\Big|_{LU_\alpha} = d\bar{B}_\alpha$ on LU_α. On LU_α, we have

$$
\begin{aligned}
(D_{\bar{H}})^2 =& (\nabla^{\mathcal{L}^B} - i_K + \bar{H})^2 && (1)\\
=& (d - i_K\bar{B}_\alpha - i_K + \bar{H})^2 && (2)\\
=& \big((d - i_K) + (d - i_K)\bar{B}_\alpha\big)^2 && (3)\\
=& \big(\exp(-\bar{B}_\alpha)(d - i_K)\exp(\bar{B}_\alpha)\big)^2 && (4)\\
=& - L_K - (L_K\bar{B}_\alpha) = -L_K, && (5)
\end{aligned}
$$

Exotic twisted equivariant cohomology of loop space

Proof.

where L_K denotes the Lie derivative of the vector field K. As the Brylinski sections are invariant, we have $L_K = L_K^{\mathcal{L}^B}$ on LU_α. So $(D_{\bar{H}})^2 = -L_K^{\mathcal{L}^B}$, which vanishes on $\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}$ as claimed. □

Notice that $D_{\bar{H}} = \nabla^{\mathcal{L}^B} - i_K + \bar{H}$ is a **flat $\mathbb{T}$-equivariant superconnection** (in the sense of Quillen) on $\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}$. Therefore $\left(\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}, D_{\bar{H}}\right)$ is a $\mathbb{Z}_2$-graded complex. We call the cohomology of this complex the **exotic twisted $\mathbb{T}$-equivariant cohomology** of loop space, denoted by $H^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$.

Completed exotic twisted equivariant cohomology of loop space

Define the **completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology** $h^*_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$ to be the cohomology of the complex $(\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}[u, u^{-1}]], \nabla^{\mathcal{L}^B} - ui_K + u^{-1}\bar{H})$.

NB the holonomy line bundle $\mathcal{L}^B$ is trivial when restricted to Z, the constant loop space, we have

Theorem (Localisation)

The restriction to the constant loops

$$res : h^*_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \cong H^*(Z, H)[u, u^{-1}]]$$

is an isomorphism.

Equivariant circle bundles - classification

This justifies the following 2 proposals:

> *RR fields in type II String Theory in a background H-flux, take values in* $\Omega^\bullet(LZ, \mathcal{L}^B)^{S^1}$ *and are closed wrt the exotic differential* $D_{\bar{H}}$. *(EOM)*

Note that the configuration space in this proposal is the much larger space of differential forms on loop space LZ, which is more naturally related to the geometric picture of (closed) strings. It also includes massive RR-fields.

Also

> *Over the rationals, D-brane charges on space-time* Z *in a background H-flux, take values in* $h^*_{\mathbb{T}}(LZ, \nabla^B : \bar{H})$.

Gerbe modules

Let $\{U_\alpha\}$ be a Brylinski cover of Z and $E = \{E_\alpha\}$ be a collection of (infinite dimensional) Hilbert bundles $E_\alpha \to U_\alpha$ whose structure group is reduced to U_{tr}, which are unitary operators on the model Hilbert space $\mathcal{H}$ of the form (identity + trace class operator). Here tr denotes the Lie algebra of trace class operators on $\mathcal{H}$.

In addition, assume that on the overlaps $U_{\alpha\beta}$ that there are isomorphisms $\phi_{\alpha\beta} : L_{\alpha\beta} \otimes E_\beta \cong E_\alpha$, which are consistently defined on triple overlaps because of the gerbe property. Then $\{E_\alpha\}$ is said to be a **gerbe module** for the gerbe $\{L_{\alpha\beta}\}$.

Gerbe modules

A **gerbe module connection** ∇^E is a collection of connections $\{\nabla^E_\alpha\}$ is of the form $\nabla^E_\alpha = d + A^E_\alpha$ where $A^E_\alpha \in \Omega^1(U_\alpha) \otimes tr$ whose curvature F^E_α on the overlaps $U_{\alpha\beta}$ satisfies

$$\phi^{-1}_{\alpha\beta}(F^E_\alpha)\phi_{\alpha\beta} = F^L_{\alpha\beta} I + F^E_\beta .$$

Using $F^L_{\alpha\beta} = B_\beta - B_\alpha$, this becomes

$$\phi^{-1}_{\alpha\beta}(B_\alpha I + F^E_\alpha)\phi_{\alpha\beta} = B_\beta I + F^E_\beta .$$

It follows that $\exp(-B)\,\mathrm{Tr}\left(\exp(-F^E) - I\right)$ is a globally well defined differential form on Z of even degree. Notice that $\mathrm{Tr}(I) = \infty$ which is why we need to consider the subtraction.

Gerbe modules Twisted K-theory

Let $E = \{E_\alpha\}$ and $E' = \{E'_\alpha\}$ be be a gerbe modules for the gerbe $\{L_{\alpha\beta}\}$. Then an element of twisted K-theory $K^0(Z, H)$ is represented by the pair (E, E'). Two such pairs (E, E') and (G, G') are equivalent if $E \oplus G' \oplus K \cong E' \oplus G \oplus K$ as gerbe modules for some gerbe module K for the gerbe $\{L_{\alpha\beta}\}$. We can assume without loss of generality that these gerbe modules E, E' are modeled on the same Hilbert space .

Suppose that $\nabla^E, \nabla^{E'}$ are gerbe module connections on the gerbe modules E, E' respectively. Then we can define the **twisted Chern character** of [BCMMS] as

$$Ch_H : K^0(Z, H) \to H^{even}(Z, H)$$

$$Ch_H(E, E') = \exp(-B)\,\mathrm{Tr}\left(\exp(-F^E) - \exp(-F^{E'})\right)$$

Path ordered exponential

Let $\mathcal{A}$ be a unital Banach algebra and $a : [0, 1] \longrightarrow \mathcal{A}$ be a continuous function. Define the **path ordered exponential**, denoted $\mathcal{T}(t) = \mathcal{T}(\exp(\int_0^1 a(s)ds))$ as the unique solution to

$$\frac{d}{dt}\mathcal{T}(t) = a(t)\mathcal{T}(t)$$
$$\mathcal{T}(0) = 1$$

Then it has a convergent power series expansion

$$\mathcal{T}(t) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} a(s_1) \cdots a(s_n) ds_1 \cdots ds_n$$

where $\Delta_n(t)$ is the n-simplex of size t, ie

$$\Delta_n(t) = \{0 \leq s_1 \leq \cdots \leq s_n \leq t\}.$$

Twisted Bismut-Chern character

Via the path ordered exponential method, lift the twisted Chern character of [BCMMS] to loop space LZ by defining $BCh_{H,\alpha}(\nabla^E, \nabla^{E'}) \in \Omega^\bullet(LU_\alpha, \mathcal{L}^B)^{\mathbb{T}}[u, u^{-1}]]$ by

$$BCh_{H,\alpha}(\nabla^E, \nabla^{E'}) =$$

$$\left(1 + \sum_{n=1}^{\infty} (-u)^{-n} \int_{\Delta_n(1)} \widehat{B}_{\alpha\, s_1} \cdots \widehat{B}_{\alpha\, s_n}\right) \Big(BCh_\alpha(\nabla^E) - BCh_\alpha(\nabla^{E'})\Big)\, \sigma_\alpha$$

$$= \mathcal{T}\left(exp\left(\frac{-1}{u}\int_0^1 \widehat{B}_{\alpha\, s} ds\right)\right) \Big(BCh_\alpha(\nabla^E) - BCh_\alpha(\nabla^{E'})\Big)\, \sigma_\alpha$$

$BCh_\alpha(\nabla^E)$ is the path ordered exponential lift of the Chern chracter to loop space LZ due to Bismut. Since the curvature of ∇^E is vector valued therefore parallel transport wrt ∇^E has to be inserted into the curvature factors before taking the trace.

Cartan model for equivariant cohomology

Define the **twisted Bismut-Chern character form** $BCh_H(\nabla^E, \nabla^{E'}) \in \Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}[u, u^{-1}]]$ to be the global form patched together from the local forms constructed above.

Theorem

(i) We have $(\nabla^{\mathcal{L}^B} - ui_K + u^{-1}\bar{H})BCh_H(\nabla^E, \nabla^{E'}) = 0$;
(ii) The exotic twisted $\mathbb{T}$*-equivariant cohomology class* $[BCh_H(\nabla^E, \nabla^{E'})]$ *does not depend on the choice of connections* $\nabla^E, \nabla^{E'}$.
(iii) One has a commutative diagram

$$K^\bullet(Z, H) \xrightarrow{BCh_H} h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$$

$$K^\bullet(Z, H) \xrightarrow{Ch_H} H^\bullet(Z, H)[u, u^{-1}]] \xleftarrow{res} h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$$

T-duality: a brief review

Consider

$$
\begin{array}{ccc}
(Z, H) & & (\widehat{Z}, \widehat{H}) \\
& {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \widehat{p}} & \\
& X &
\end{array}
$$

where $Z, \widehat{Z}$ are principal circle bundles over a base X with fluxes H and $\widehat{H}$, respectively, satisfying $p_*(H) = c_1(\widehat{Z})$, $\widehat{p}_*(\widehat{H}) = c_1(Z)$ and $H - \widehat{H}$ is exact on the correspondence space $Z \times_X \widehat{Z}$. The T-duality Theorem for circle bundles in [BEM] states that there is an isomorphism of twisted K-theories $K^\bullet(Z, H) \cong K^{\bullet+1}(\widehat{Z}, \widehat{H})$ and an isomorphism of twisted cohomology theories, $H^\bullet(Z, H) \cong H^{\bullet+1}(\widehat{Z}, \widehat{H})$,

T-duality: a loop space perspective

As a consequence of our

1. Localisation Theorem,
2. properties of the twisted Bismut-Chern character,
3. the T-duality Theorem for circle bundles,

we obtain a T-duality isomorphism on loop space as below.

T-duality: a loop space perspective

Theorem (T-duality : a loop space perspective)

In the notation above, there is an isomorphism

$$T : h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \xrightarrow{\cong} h^{\bullet+1}_{\mathbb{T}}(L\widehat{Z}, \nabla^{\mathcal{L}^{\widehat{B}}} : \bar{\widehat{H}}),$$

such that the following diagram commutes,

$$\begin{array}{ccc}
K^\bullet(Z, H) & \xrightarrow[\cong]{T} & K^{\bullet+1}(\hat{Z}, \hat{H}) \\
\downarrow BCh_H & & \downarrow BCh_{\hat{H}} \\
h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) & \xrightarrow[T]{} & h^{\bullet+1}_{\mathbb{T}}(L\widehat{Z}, \nabla^{\mathcal{L}^{\widehat{B}}} : \bar{\widehat{H}}) \\
res \downarrow \cong & & \cong \downarrow res \\
H^\bullet(Z, H)[u, u^{-1}]] & \xrightarrow[T]{\cong} & H^{+1}(\widehat{Z}, \widehat{H})[u, u^{-1}]]
\end{array}$$

with curved arrows $Ch_H : K^\bullet(Z, H) \to H^\bullet(Z, H)[u, u^{-1}]]$ and $Ch_{\hat{H}} : K^{\bullet+1}(\hat{Z}, \hat{H}) \to H^{+1}(\widehat{Z}, \widehat{H})[u, u^{-1}]]$.

(6)