

Graded T-duality with H-flux for $2d$ σ -models

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joint with Varghese Mathai :

F. Han and V. Mathai, *T-duality with H-flux for 2d σ -models*,
arXiv :2207.03134.

Developed from and motivated by the following papers as well as the papers listed in the Reference :

1. [A] M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, Astérisque, 131 (1985), 43-59.
2. [B] J-M. Bismut, *Index theorem and equivariant cohomology on the loop space*, Comm. Math. Phys. 98 (1985), no. 2, 213-237.
3. [BEM] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality : Topology Change from H-flux*, Comm. Math. Phys. 249 (2004), 383-415.

Topological T-duality with H-flux

Consider such a pair :

$$\begin{array}{ccc}
 (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\
 \searrow \pi & & \swarrow \widehat{\pi} \\
 & X &
 \end{array}$$

Z and \widehat{Z} are principal circle bundles over a manifold X ;

A and \widehat{A} are connections on them respectively;

H is a background flux, i.e. a closed 3-form on Z with \mathbb{Z} periods, and similarly for \widehat{H}

subject to the following relations :

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^A,$$

F^A and $F^{\widehat{A}}$ are curvatures of A and \widehat{A} respectively.

Topological T-duality with H-flux

$$\begin{array}{ccc}
 (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\
 \searrow \pi & & \swarrow \widehat{\pi} \\
 & X & \\
 \pi_!(H) = F^{\widehat{A}}, & & \widehat{\pi}_!(\widehat{H}) = F^A
 \end{array}$$

the model of [topological T-duality with H-flux](#) formulated by Bouwknegt-Evslin-Mathai.

More generally, there is the model of [pair of principal torus bundles](#) by Bouwknegt-Hannabuss-Mathai.

Topological T-duality with H-flux

Closely related to the *Strominger-Yau-Zaslow Conjecture* concerning realization of Calabi–Yau manifolds and their mirrors as torus bundles over same base, proposed in their paper *Mirror Symmetry is T-duality*.

Topological T-duality with H-flux

For the T-dual pair with H-flux,

$$\begin{array}{ccc}
 (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\
 \searrow \pi & & \swarrow \widehat{\pi} \\
 & X &
 \end{array}$$

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^A$$

several dual results, to just name a few, have been proved, verify or coincide with predictions of physicists :

Topological T-duality with H-flux

(1) Twisted cohomology

Let M be a smooth manifold, ω a closed 3 form on M . Consider the twisted de Rham complex

$$(\Omega^*(M), d + \omega)$$

and the cohomology of this complex $H^*(M, \omega)$. They are \mathbb{Z}_2 -graded.

In the case of $M = Z$ and $\omega = H$,

$G \in \Omega^\bullet(Z)^\mathbb{T}$, the total RR fieldstrength,

$$\begin{aligned} G &\in \Omega^{even}(Z)^\mathbb{T} && \text{for Type IIA;} \\ G &\in \Omega^{odd}(Z)^\mathbb{T} && \text{for Type IIB.} \end{aligned}$$

Topological T-duality with H-flux

To relate the Z and \widehat{Z} sides, a fundamental construction is the **correspondence space**

$$\begin{array}{ccc}
 (Z \times_X \widehat{Z}, [p^* H] = [\widehat{p}^* \widehat{H}]) & & \\
 \swarrow p & & \searrow \widehat{p} \\
 (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\
 \searrow \pi & & \swarrow \widehat{\pi} \\
 X & &
 \end{array}$$

Topological T-duality with H-flux

In [BEM], it shows that the **Hori map**

$$(1) \quad T_H G = \int_{\mathbb{T}} e^{A \wedge \hat{A}} G,$$

a **Fermionic Fourier transformation** through the correspondence space, gives

$$T_H: \Omega^{\bar{k}}(Z)^{\mathbb{T}} \rightarrow \Omega^{\bar{k}+1}(\hat{Z})^{\hat{\mathbb{T}}},$$

for $k = 0, 1$, (where \bar{k} denotes the parity of k) is isomorphism, inducing isomorphism on twisted cohomology groups,

$$T_H: H^*(Z, H) \xrightarrow{\cong} H^{*+1}(\hat{Z}, \hat{H}).$$

Topological T-duality with H-flux

(2) Twisted K-theory

Let M be a smooth manifold, ω a closed 3 form on M with integral period. Then one has the twisted K -theory $K(M, \omega)$.

There were vast development of twisted K -theory by the work of [Donovan-Karoubi](#), [Atiyah-Segal](#), [Freed-Hopkins-Teleman](#).

The [twisted Chern classes](#) for twisted K -theory have been studied by [Atiyah-Segal](#) (using Atiyah-Hirzebruch spectral sequence), [Bouwknegt-Carey-Mathai-Murray-Stevenson](#) (Chern-Weil theory).

Quantization of the twisted Chern class lead to the [Mathai-Melrose-Singer *Fractional Index Theory*](#) in the torsion case.

In particular, there is a [twisted Chern character](#) map :

$$\mathrm{Ch}_\omega : K^*(M, \omega) \rightarrow H^*(M, \omega)$$

D-brane charges in Type IIA String theory are classified by twisted K-theory $K^0(Z, H)$ and in Type IIB String theory are classified by twisted K-theory $K^1(Z, H)$ (Bouwknegt-Mathai 2000, Bouwknegt-Carey-Mathai-Murray-Stevenson 2002)

In [BEM], using the correspondence space, it shows that there is an isomorphism

$$T_K: K^*(Z, H) \rightarrow K^{*+1}(\hat{Z}, \hat{H}),$$

and moreover, there is commutative diagram,

$$(2) \quad \begin{array}{ccc} K^*(Z, H) & \xrightarrow{T_K} & K^{*+1}(\hat{Z}, \hat{H}) \\ \text{Ch}_H \downarrow & & \downarrow \text{Ch}_{\hat{H}} \\ H^*(Z, H) & \xrightarrow{T_H} & H^{*+1}(\hat{Z}, \hat{H}) \end{array}$$

Topological T-duality with H-flux

(3) Loop space perspective

Atiyah-Witten-Bismut's work ([A], [B]) studied equivariant cohomology of free loop spaces, and formally realized the Atiyah-Singer index theory as fixed point theory on free loop spaces.

Indicates that T-duality with H -flux and the Hori formulae for spacetime should be a shadow of T-duality and Hori formulae for loop space of spacetime.

Along this free loop space perspective, we mention some work :

A. Linshaw and V. Mathai, *Twisted Chiral De Rham Complex, Generalized Geometry, and T-duality*, Comm. Math. Phys., 339, No. 2, (2015).

F. Han and V. Mathai, *Exotic twisted equivariant cohomology of loop spaces, twisted Bismut-Chern character and T-duality*, Comm. Math. Phys., 337, no. 1, (2015) 127–150.

Topological T-duality with H-flux

(4) Double loop spaces perspective

The main topic for this talk.

Double loop the T-dual pair, we have the following picture :

$$\begin{array}{ccc}
 (\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) & & (\mathcal{L}_{\hat{H},\tau}, \nabla^{\mathcal{L}_{\hat{H},\tau}}) \\
 \downarrow & & \downarrow \\
 C^\infty(T^2, Z)^H & & C^\infty(T^2, \hat{Z})^{\hat{H}} \\
 \searrow^{LL\pi} & & \swarrow^{LL\hat{\pi}} \\
 & C^\infty(T^2, X) &
 \end{array}$$

where T^2 is the 2-dimensional torus, $C^\infty(T^2, Z) = LLZ$ is the double loop space, $C^\infty(T^2, Z)^H$ is certain circle bundle over the double loop space, τ is a modular parameter, $(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}})$ is the **average τ -holonomy line bundle** with a canonical connection. Similarly notations on the dual side.

Topological T-duality with H-flux

We will explain the notations in more details later, construct some complexes from the objects $(\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) \rightarrow C^\infty(T^2, Z)^H$ and establish the T-dual map between the Z side and \widehat{Z} side.

This result in some sense is the T -duality with H -flux for $2d$ σ -model, answering a question of Hori.

Double loop spaces

To study T-duality from the perspective of 2d σ -models, joint with Mathai, we give relevant constructions and discover some properties on double loop spaces.

A. we introduce the double loop Brylinski cover for M :
 $\{U_\alpha\}$: maximal open cover of M with the property that $H^i(U_{\alpha_I}) = 0$ for $i \geq 3$, where $U_{\alpha_I} = \bigcap_{i \in I} U_{\alpha_i}$, $|I| < \infty$.

In fact, let $x: T^2 \rightarrow M$ be a smooth loop in M and U_x a tubular neighbourhood of x in M . $\{LLU_x, x \in LLM\}$ covers LLM .

Double loop spaces

B. we construct various transgression maps or averaging maps :

Let ev is the evaluation map

$$ev : LLM \times T^2 \rightarrow M : (x, s, t) \mapsto x(s, t),$$

we have the **double transgression** map :

$$\mu_{1,2} : \Omega^\bullet(U_{\alpha_I}) \longrightarrow \Omega^{\bullet-2}(LLU_{\alpha_I})$$

defined by

$$\mu_{1,2}(\xi_I) = \int_{T^2} ev^*(\xi_I), \quad \xi_I \in \Omega^\bullet(U_{\alpha_I}).$$

Double loop spaces

we have the [averaging after transgression](#) map :

$$\overline{\mu}_1^2 : \Omega^\bullet(\mathbf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LL\mathbf{U}_{\alpha_I})$$

defined by

$$\overline{\mu}_1^2(\xi_I) = \int_{S^1} \left(\int_{S^1} ev^*(\xi_I) \right) dt, \quad \xi_I \in \Omega^\bullet(\mathbf{U}_{\alpha_I}),$$

i.e. integrate $ev^*(\xi_I)$ along the first circle and then average along the second circle. Similarly, one has

$$\overline{\mu}_2^1 : \Omega^\bullet(\mathbf{U}_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LL\mathbf{U}_{\alpha_I}).$$

Double loop spaces

Let $\omega \in \Omega^i(M)$. One also has the **double loop averaging** map

$$\bar{\bar{\omega}} := \int_{T^2} ev^*(\omega) ds \wedge dt \in \Omega^i(LLM).$$

Clearly $L_{K_i}\bar{\bar{\omega}} = 0$, $i = 1, 2$. Moreover it is not hard to see that

$$d\bar{\bar{\omega}} = \bar{\bar{d\omega}}, \quad \mu_{1,2}(\omega) = \iota_{K_2}\iota_{K_1}\bar{\bar{\omega}}.$$

In addition to evaluation map (3), there are also **partial evaluation maps**


$$ev_1 : LLM \times S^1 \rightarrow LM, \quad (x, s) \mapsto x(s, *),$$

$$ev_2 : LLM \times S^1 \rightarrow LM, \quad (x, t) \mapsto x(*, t),$$

and certain projections from double loops to one loops

$$\pi_i : LLM \rightarrow LM, \quad i = 1, 2$$

defined by $\pi_1 = ev_2|_{t=0}$, i.e restriction to the first circle and

$\pi_2 = ev_1|_{s=0}$, i.e restriction to the second circle. 

Double loop spaces

C. we construct the **average holonomy line bundle** on the double loop space arising from the following data.

Suppose M carries a gerbe with connection $(H, B_\alpha, F_{\alpha\beta}, (L_{\alpha\beta}, \nabla^{L_{\alpha\beta}}))$, with $H \in \Omega^3(M)$, $B_\alpha \in \Omega^2(U_\alpha)$ and $(L_{\alpha\beta}, \nabla^{L_{\alpha\beta}})$ being a complex line bundle over $U_{\alpha\beta} = U_\alpha \cap U_\beta$ such that

$$H = dB_\alpha \text{ on } U_\alpha,$$

$$B_\beta - B_\alpha = F_{\alpha\beta} = (\nabla^{L_{\alpha\beta}})^2 \text{ on } U_\alpha \cap U_\beta,$$

$$(L_{\alpha\beta}, \nabla^{L_{\alpha\beta}}) \otimes (L_{\beta\gamma}, \nabla^{L_{\beta\gamma}}) \otimes (L_{\gamma\alpha}, \nabla^{L_{\gamma\alpha}}) \simeq (\mathbb{C}, d) \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Double loop spaces

Let \mathcal{L} be the holonomy line bundle on LM arising from H .

Let $\tau \in \mathbb{H}$, the upper half plane. This means now we consider the complex structures on the source $T^2 \rightarrow M$.

Roughly speaking, our construction of the τ -average holonomy line bundle is to make sense of the following line bundle over LLM

$$\pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L})^{\otimes \tau},$$

i.e. $\pi_1^*(\mathcal{L}) \otimes$ tensored with the “ τ -th power” of $\pi_2^*(\mathcal{L})^{\otimes \tau}$.

Unfortunately, since τ is not an integer, but a complex number with positive imaginary part, $\pi_2^*(\mathcal{L})^{\otimes \tau}$ does not make sense.

Double loop spaces

In the following, we will explain a situation that can make sense out of this. Let ξ be a complex line bundle over a manifold X .

Let $\mathfrak{U} = \{U_\alpha\}$ be an open good cover of X . Let $\{g_{\alpha\beta}\}$ be a system of $U(1)$ -valued transition functions w.r.t \mathfrak{U} . This gives us a closed Čech cocycle $\{\theta_{\alpha\beta}\}$ valued in \mathbb{R}/\mathbb{Z} by taking $\theta_{\alpha\beta} = \frac{1}{2\pi i} \ln g_{\alpha\beta}$ in the argument interval $[0, 2\pi)$. So $\{\theta_{\alpha\beta}\} \in C^1(\mathfrak{U}, \mathbb{R}/\mathbb{Z})$.

Double loop spaces

Let

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$$

be the obvious exact sequence. It is not hard to show that

(1) $\{\theta_{\alpha\beta}\}$ can be lifted as the image of a Čech cocycle in $C^1(\mathcal{U}, \mathbb{R}) \iff \xi$ is trivial.

(2) Different liftings differ by δc for some $c \in C^1(\mathcal{U}, \mathbb{Z})$.

Let ξ be trivial and $\eta_{\alpha\beta} \in C^1(\mathcal{U}, \mathbb{R})$ be a lifting of $\{\theta_{\alpha\beta}\}$. Then we can consider the \mathbb{C} -valued functions $\{e^{2\pi i\tau\eta_{\alpha\beta}}\}$, which satisfy

$$e^{2\pi i\tau\eta_{\alpha\beta}} \cdot e^{2\pi i\tau\eta_{\beta\gamma}} \cdot e^{2\pi i\tau\eta_{\gamma\alpha}} = 1$$

and therefore glues us a complex line bundle over X . One can consider it as the τ -th power of ξ . One may think of this construction by taking $\tau = \frac{1}{n}$, $n \in \mathbb{Z}$ and the construction of an n -th root of ξ .

Double loop spaces

Let us come back to the double loop space. Although $\pi_2^*(\mathcal{L})^{\otimes \tau}$ does not make sense, but suppose

$$p: \mathcal{S} \rightarrow LLM$$

be the circle bundle of $\pi_2^*(\mathcal{L})$, the pull back $p^*(\pi_2^*(\mathcal{L}))$ is a trivial bundle over \mathcal{S} . Then $p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}$ makes sense as explained above. So on \mathcal{S} , we have the line bundle

$$p^*(\pi_1^*(\mathcal{L})) \otimes p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}.$$

This is just the τ -average holonomy line bundle we are going to construct.

Double loop spaces

Note that on LLM , there is the T^2 -action. Hence we have to give a T^2 -invariant transition function for the line bundle

$$p^*(\pi_1^*(\mathcal{L})) \otimes p^*(\pi_2^*(\mathcal{L}))^{\otimes \tau}.$$

To achieve this, we have to give $(K_1 + \tau K_2)$ -invariant transition functions for this line bundle, which we will explain in the following.

Double loop spaces

For any double loop $x \in LLU_\alpha \cap LLU_\beta$, i.e. $x: T^2 \rightarrow U_{\alpha\beta}$, denote the holonomy of the $\nabla^{L_{\alpha\beta}}$ along the K_1 -direction of by hol^1 , which is a function of t ; and the holonomy of the $\nabla^{L_{\alpha\beta}}$ along the K_2 -direction by hol^2 , which is a function of s .

Consider the function on $LLU_\alpha \cap LLU_\beta$

$$g_{\alpha\beta} := e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1}.$$

Note here for $\ln hol^1$, it is continuously defined for $t \in [0, 1)$, and for $\ln hol^2$, it is continuously defined for $s \in [0, 1)$.

Double loop spaces

$$\begin{aligned}
& L_{K_1 + \tau K_2} g_{\alpha\beta} \\
&= L_{K_1 + \tau K_2} \left(e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1} \right) \\
&= e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1} [L_{\tau K_2} \overline{\ln hol^1_{\alpha\beta}}^{-2} + L_{K_1} \tau \overline{\ln hol^2_{\alpha\beta}}^1] \\
&= e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1} [\tau K_2 \overline{\ln hol^1_{\alpha\beta}}^{-2} + \tau K_1 \overline{\ln hol^2_{\alpha\beta}}^1] \\
&= e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1} \tau [\overline{\iota_{K_2} d \ln hol^1_{\alpha\beta}}^{-2} + \overline{\iota_{K_1} d \ln hol^2_{\alpha\beta}}^1] \\
&= e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1} \tau \left[\overline{\iota_{K_2} \iota_{K_1} \overline{F_{\alpha\beta}}^1}^{-2} + \overline{\iota_{K_1} \iota_{K_2} \overline{F_{\alpha\beta}}^2}^1 \right] \\
&= e^{\overline{\ln hol^1_{\alpha\beta}}^{-2}} \cdot e^{\tau \overline{\ln hol^2_{\alpha\beta}}^1} \tau \left[\overline{\iota_{K_1} \iota_{K_2} \overline{\overline{F_{\alpha\beta}}}} + \overline{\iota_{K_2} \iota_{K_1} \overline{\overline{F_{\alpha\beta}}}} \right] \\
&= 0.
\end{aligned}$$

So $g_{\alpha\beta}$ is $(K_1 + \tau K_2)$ -invariant.

Double loop spaces

Denote

$$h_{\alpha\beta\gamma} := g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}.$$

On the triple intersection $LLU_{\alpha} \cap LLU_{\beta} \cap LLU_{\gamma}$,

$$\text{hol}_{\alpha\beta}^i \text{hol}_{\beta\gamma}^i \text{hol}_{\gamma\alpha}^i = 1, \quad i = 1, 2.$$

Hence $\ln \text{hol}_{\alpha\beta}^i + \ln \text{hol}_{\beta\gamma}^i + \ln \text{hol}_{\gamma\alpha}^i \in 2\pi i\mathbb{Z}$. As $\ln \text{hol}$'s are continuously defined, one must have

$$(3) \quad h_{\alpha\beta\gamma} = e^{2\pi i m_{\alpha\beta\gamma} \tau}$$

for some $m_{\alpha\beta\gamma} \in \mathbb{Z}$, where $\{m_{\alpha\beta\gamma}\}$ forms the Čech cocycle representing $\pi_2^*(c_1(\mathcal{L}_B))$ in $H^2(LLM, \mathbb{Z})$ with $c_1(\mathcal{L}_B)$ being the first Chern class of the holonomy line bundle \mathcal{L}_B on LM arising from the ω on M .

Double loop spaces

Let $p : \mathcal{S}_B \rightarrow LM$ be the circle bundle of the line bundle $\mathcal{L}_B \rightarrow LM$. Then $p^*\mathcal{L}_B$ is a trivial line bundle over \mathcal{S}_B . Hence the class $p^*(c_1(\mathcal{L}_B))$ is 0 on \mathcal{S}_B . Therefore $\tilde{\pi}_2^* \circ p^*(c_1(\mathcal{L}_B))$ is 0 on $p^*\mathcal{S}_B$, the pulled back circle bundle over LLM .

$$(4) \quad \begin{array}{ccc} p^*\mathcal{S}_B & \xrightarrow{\tilde{\pi}_2} & \mathcal{S}_B \\ \tilde{p} \downarrow & & \downarrow p \\ LLM & \xrightarrow{\pi_2} & LM \end{array}$$

For simplicity, in the sequel we will denote the total space $p^*\mathcal{S}_B$ by LLM^ω , which carries the induced T^2 -action arising from LLM .

Double loop spaces

Using $\{g_{\alpha\beta}\}$, after some treatment on LLM^ω , we can construct a complex line bundle $\mathcal{L}_{B,\tau}$ on LLM^ω , which we call **average τ -holonomy line bundle**, which also carries a canonical connection $\nabla^{\mathcal{L}_{B,\tau}}$ out from the B -field.

Double loop spaces

D. Equivariantly super flatness

Denote by $\Omega_{bas}^\bullet(LLM^\omega, \mathcal{L}_{H,\tau})$ the space of basic differential forms on LLM^ω with values in the average τ -holonomy line bundle $\mathcal{L}_{B,\tau}$. Here basic form means that contracted with vertical tangent vectors gives 0. Let u be an indeterminate such that $\deg u = 2$. Consider the odd operator

$$(5) \quad Q_{B,\tau} := \nabla^{\mathcal{L}_{B,\tau}} - u_{K_1 + \tau K_2} + u^{-1} \tilde{p}^* \bar{\omega}$$

which acts on $\Omega^\bullet(LLM^\omega, \mathcal{L}_{B,\tau})_{bas}[[u, u^{-1}]]$.

Double loop spaces

Theorem (H.-Mathai, 2022)

The following identities hold,

$$\frac{1}{2}[Q_{B,\tau}, Q_{B,\tau}] = Q_{B,\tau}^2 = -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}, \quad [Q_{B,\tau}, -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}] = 0,$$

where $L_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}$ is the Lie derivative along the direction $K_1 + \tau K_2$.

So the odd operator $Q = Q_{B,\tau}$ and the even operator $P = -uL_{K_1+\tau K_2}^{\mathcal{L}_{B,\tau}}$ obey the relations

$$\frac{1}{2}[Q, Q] = P, \quad [Q, P] = 0$$

of the superalgebra considered in Witten's paper *Supersymmetry and Morse theory*.

Double loop spaces

E. Localization

The above theorem tells us there is a complex

$$(\Omega_{bas}^\bullet(LLM^\omega, \mathcal{L}_{B,\tau})^{K_1+\tau K_2}[[u, u^{-1}]], Q_{B,\tau}).$$

Note that the zeros of the complex vector field $K_1 + \tau K_2$ are T^2 -fixed points of LLM , i.e. M . We have the Borel-Witten type localization :

Theorem (H.-Mathai, 2022)

Let $i: M \times S^1 \rightarrow LLM^\omega$ be the inclusion map. Then the restriction map

$$\begin{aligned} i^* : & (\Omega_{bas}^\bullet(LLM^\omega, \mathcal{L}_{B,\tau})^{K_1+\tau K_2}[[u, u^{-1}]], Q_{B,\tau}) \\ & \rightarrow (\Omega_{bas}^\bullet(M \times S^1)[[u, u^{-1}]], d + u^{-1}p^*\omega) \cong (\Omega^\bullet(M)[[u, u^{-1}]], d + u^{-1}\omega) \end{aligned}$$

is a quasi-isomorphism, $\forall \tau \in \mathbb{H}$.

Now we apply our theory on double loop spaces to study T-duality from the perspective of 2d σ -models.

Recall that we have the T-dual pair with H-flux :

$$\begin{array}{ccc} (Z, A, H) & & (\widehat{Z}, \widehat{A}, \widehat{H}) \\ & \searrow \pi & \swarrow \widehat{\pi} \\ & X & \end{array}$$

$$\pi_!(H) = F^{\widehat{A}}, \quad \widehat{\pi}_!(\widehat{H}) = F^A$$

Double loop the T-dual pair, we have the following picture :

$$\begin{array}{ccc} (\mathcal{L}_{H,\tau}, \nabla^{\mathcal{L}_{H,\tau}}) & & (\mathcal{L}_{\widehat{H},\tau}, \nabla^{\mathcal{L}_{\widehat{H},\tau}}) \\ \downarrow & & \downarrow \\ C^\infty(T^2, Z)^H & & C^\infty(T^2, \widehat{Z})^{\widehat{H}} \\ & \searrow LL\pi & \swarrow LL\widehat{\pi} \\ & C^\infty(T^2, X) & \end{array}$$

A sheaf over upper half plane \mathbb{H}

For the pair (Z, H) , define a sheaf $(G(C^\infty(T^2, Z)^H, \mathcal{L}_H), \mathcal{Q}_H)$ on \mathbb{H} of commutative differential graded algebras that to $U \subset \mathbb{H}$ assigns the graded complex of $\mathcal{O}(U)$ -modules

$$\begin{aligned} & (G(C^\infty(T^2, Z)^H, \mathcal{L}_H)(U), \mathcal{Q}_H) \\ & := \bigoplus_{m \in \mathbb{Z}} \left(\mathcal{O}(U; \Omega_{bas}^{\bullet, \mathbb{T}}(C^\infty(T^2, Z)^H, \mathcal{L}_{H, \tau}^{\otimes m})[[u, u^{-1}]]^{K_1 + \tau K_2}) \cdot y^m, Q_{mB, \tau} \right), \end{aligned}$$

where $\mathcal{O}(U; \Omega_{bas}^{\bullet, \mathbb{T}}(C^\infty(T^2, Z)^H, \mathcal{L}_{H, \tau}^{\otimes m})[[u, u^{-1}]]^{K_1 + \tau K_2})$ means i.e. for each $\tau \in U$, one assigns to it an element in $\Omega_{bas}^{\bullet, \mathbb{T}}(C^\infty(T^2, Z)^H, \mathcal{L}_{H, \tau}^{\otimes m})[[u, u^{-1}]]^{K_1 + \tau K_2}$. Dually, one can also define the sheaf $(G(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}), \mathcal{Q}_{\widehat{H}})$.

Note that we have assembled mH for $m \in \mathbb{Z}$, i.e. we consider the graded version.

Passing to cohomology, we get the sheaves $\mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H)$ and $\mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}})$. The localisation theorem tells us that the restriction maps

$$res : \mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H) \rightarrow \mathcal{G}(Z, H), \quad \widehat{res} : \mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \rightarrow \mathcal{G}(\widehat{Z}, \widehat{H})$$

are isomorphisms of sheaves.

In a previous work joint with Mathai, we constructed the **graded Hori morphisms** between the sheaves

$$GHor_* : (G(Z, H), D^H) \rightarrow (G(\widehat{Z}, \widehat{H}), D^{\widehat{H}}), \quad GHor : \mathcal{G}(Z, H) \rightarrow \mathcal{G}(\widehat{Z}, \widehat{H}).$$

$$\widehat{GHor}_* : (G(\widehat{Z}, \widehat{H}), D^{\widehat{H}}) \rightarrow (G(Z, H), D^H), \quad \widehat{GHor} : \mathcal{G}(\widehat{Z}, \widehat{H}) \rightarrow \mathcal{G}(Z, H).$$

and showed that **they send Jacobi forms to Jacobi forms**.

Now we construct the **graded Hori morphisms for 2d σ -models** by

$$GHor^\sigma := \widehat{res}^{-1} \circ GHor \circ res : \mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H) \rightarrow \mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}),$$

$$\widehat{GHor}^\sigma := res^{-1} \circ \widehat{GHor} \circ \widehat{res} : \mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \rightarrow \mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H),$$

assembled in the following commutative diagram,

$$\begin{array}{ccc} \mathcal{G}(C^\infty(T^2, Z)^H, \mathcal{L}_H) & \begin{array}{c} \xrightarrow{GHor^\sigma} \\ \xleftarrow{\widehat{GHor}^\sigma} \end{array} & \mathcal{G}(C^\infty(T^2, \widehat{Z})^{\widehat{H}}, \mathcal{L}_{\widehat{H}}) \\ \downarrow \text{res} \cong & & \downarrow \widehat{res} \cong \\ \mathcal{G}(Z, H) & \begin{array}{c} \xrightarrow{GHor} \\ \xleftarrow{\widehat{GHor}} \end{array} & \mathcal{G}(\widehat{Z}, \widehat{H}) \end{array}$$

Theorem (H.-Mathai, 2022)

One has

$$(6) \quad \widehat{GHor}^\sigma \circ GHor^\sigma = -y \frac{\partial}{\partial y}, \quad GHor^\sigma \circ \widehat{GHor}^\sigma = -y \frac{\partial}{\partial y}.$$

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