Geometric and Non-Geometric T-duality with Higher Bundles

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Geometric string background:

- A (Riemannian) manifold X
- A principal/affine torus bundle $\pi : P \to X$ (with connection)
- An abelian gerbe (with connection) $\mathscr{G}$ on the total space of P
- No equations of motion imposed.

Topological T-duality:

$$\begin{array}{ccccccc} \check{\mathscr{G}} \longrightarrow \check{P} & & & & \hat{P} \longleftarrow \hat{\mathscr{G}} \\ & \overset{\check{\pi}}{\searrow} & & \overset{\hat{\pi}}{\swarrow} & \\ & & X & & \end{array}$$

From exactness of the Gysin sequence

$$\ldots \to \mathrm{H}^3(X,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(P,\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{F\cup} \mathrm{H}^4(X,\mathbb{Z}) \to \ldots$$

$$(F,H) = (\pi_*\hat{H}, H) \longleftrightarrow (\hat{F},\hat{H}) = (\pi_* H, \hat{H})$$

Bouwknegt, Evslin, Hannabuss, Mathai (2004)

Topological T-duality, geometrically

T-correspondence:

$$
\begin{array}{ccccccc}
 & & & \mathscr{G}_{\mathrm{C}} = \check{\mathsf{p}}^*\check{\mathscr{G}} \otimes \hat{\mathsf{p}}^*\hat{\mathscr{G}}^{-1} \cong \mathcal{I} & & & \\
 & & & \downarrow & & & \\
 & & & \check{P} \times_X \hat{P} & & & \\
 & & \swarrow^{\check{\mathsf{p}}} & & \searrow^{\hat{\mathsf{p}}} & & \\
\check{\mathscr{G}} & \longrightarrow & \check{P} & & \hat{P} & \longleftarrow & \hat{\mathscr{G}} \\
 & & \searrow_{\check{\pi}} & & \swarrow_{\hat{\pi}} & & \\
 & & & X & & &
\end{array}
$$

Bunke, Rumpf, Schick (2005, 2006)

Principal 2-bundles (without connections) over X:

$$
\begin{array}{ccc}
 & \mathscr{P}_{\mathrm{C}} & \\
\swarrow^{\check{\mathsf{p}}} & & \searrow^{\hat{\mathsf{p}}} \\
\check{\mathscr{P}} & & \hat{\mathscr{P}}
\end{array}
$$

Nikolaus, Waldorf (2018)

Two open problems

I. What about differential refinement beyond just topology?

II. What about non-geometric backgrounds?

F^3: H has no legs along fiber
T-duality: identity

F^2: H has 1 leg along fiber
T-duality $\rightarrow$ geometric string background

F^1: H has 2 legs along fiber
T-duality $\rightarrow$ Q-space, (e.g. T-folds) locally geometric

F^0: H has all legs along fiber
T-duality $\rightarrow$ R-space, non-geometric

Nikolaus/Waldorf: only $F^2 \leftrightarrow F^2$ and $F^2 \leftrightarrow F^1$ T-dualities

Why is this interesting/hard?

I. Connections on principal 2-bundles often require adjustment

II. Need to use suitable groupoids and augmented groupoids

Outline

- Adjusted connections on principal 2-bundles
  - Derivation
  - Some examples: higher instantons, gauged SUGRA
- Geometric T-duality with principal 2-bundles
- Everything explicit, fully explicit examples
- Non-geometric T-dualities:
  - Q-spaces
  - R-spaces

Not an expert in T-duality, both questions and comments welcome!

Principal 2-bundles or Non-Abelian Gerbes

with Adjusted Connections

Quick recap: Higher principal bundles

- Principal bundles with connections: 1d parallel transport
- Higher-dimensional parallel transport: higher geometry
  - Higher/categorified gauge group $\mathscr{G}$
  - Higher/categorified spaces: higher groupoids
  - Connections containing higher degree form fields
- Cocycle description of principal $\mathscr{G}$-bundle over manifold X:
  - Surjective submersion $\sigma : Y \twoheadrightarrow X$, e.g. $Y = \sqcup_a U_a$
  - Čech groupoid:

$$\check{\mathscr{C}}(\sigma) \;:\; Y \times_X Y \rightrightarrows Y \;, \quad (y_1, y_2) \circ (y_2, y_3) = (y_1, y_3) \;.$$

  - Bundle: Functor $g : \check{\mathscr{C}}(\sigma) \to \mathsf{B}\mathscr{G}$
  - Equivalences/bundle isomorphisms: natural isomorphisms.

Connections on principal 2-bundles: work a bit more...

Breen, Messing (2005), Aschieri, Cantini, Jurčo (2005)

Data obtained for 2-group $\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G}$ and Lie 2-algebra $\mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$:

$$h \in \Omega^0(Y^{[3]}, \mathsf{H}) \quad \Lambda \in \Omega^1(Y^{[2]}, \mathfrak{h}) \quad B \in \Omega^2(Y, \mathfrak{h}) \quad \delta \in \Omega^2(Y^{[2]}, \mathfrak{h})$$

$$g \in \Omega^0(Y^{[2]}, \mathsf{G}) \quad A \in \Omega^1(Y, \mathfrak{g})$$

- Note that δ sticks out unnaturally.
- It was dropped in most later work (Baez, Schreiber, ...)
- Price to pay: part of curvature must vanish

Can't live with or without fake curvature?

$$\mathcal{F} := \mathrm{d}A + \tfrac{1}{2}[A,A] + \mathsf{t}(B) \overset{!}{=} 0$$

Without this condition:

- Higher parallel transport is not reparameterization invariant
- Closure of gauge transformations and composition of cocycles:

$$(a_{23}^{-1} a_{12}^{-1}) \rhd (m_{123}^{-1}(\mathcal{F}_1 \rhd m_{123})) \overset{!}{=} 0$$

- 6d Self-duality equation $H = \star H$ is not gauge-covariant:

$$H \rightarrow \tilde{H} = g \rhd H - \mathcal{F} \rhd \Lambda$$

With this condition:

- Principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is flat principal G-bundle.
- Higher connections are locally abelian!

Gastel (2019), CS, Schmidt (2020)

- Reason for lack of popularity of gerbes in string theory(?)

Solution: Adjustment

Many (all?) higher gauge groups come with

Adjustment of 2-group $\mathcal{G}$: CS, Schmidt (2020), Rist, CS, Wolf (2022)

Map $\kappa : \mathcal{G} \times \mathsf{Lie}(\mathcal{G}) \to \mathsf{Lie}(\mathcal{G})$ of degree -1 such that

$$(g_2^{-1} g_1^{-1}) \rhd (h^{-1}(X \rhd h)) + g_2^{-1} \rhd \kappa(g_1, X) \\ + \kappa(g_2, g_1^{-1} X g_1 - \mathsf{t}(\kappa(g_1, X))) - \kappa(\mathsf{t}(h) g_1 g_2, X) = 0$$

for all $g_{1,2} \in \mathcal{G}_0$ and $X \in \mathsf{Lie}(\mathcal{G})_0$.

Remarks:

- Adjustment is additional algebraic datum
- Necessary for consistent definition of invariant polynomials.
- Specifies $\delta \in \Omega^2(Y^{[2]}, \mathfrak{h})$ in terms of g and F
- Adjustment of curvature/cocycle/coboundary relations
- Can drop fake flatness condition, all problems go away

Example: Higher Instantons

"Fundamental" $\mathsf{SU}(2)$-instanton

- Higher dim. Hopf fibration: principal $\mathsf{SU}(2)$-bundle: $S^7 \to S^4$
- "Doubled" to $\mathsf{Spin}(4)$-bundle $\mathsf{Spin}(5) \to \mathsf{Spin}(5)/\mathsf{Spin}(4) \cong S^4$
- Note: with indefinite metric, 2nd Chern number vanishes

"Higher instanton/monopole/sd string" Rist, CS, Wolf (2022)

- Note: $\mathsf{String}(5)/\mathsf{String}(4) \cong (S^4 \rightrightarrows S^4)$
- Have principal 2-bundle $\mathsf{String}(5) \to \mathsf{String}(5)/\mathsf{String}(4) \cong S^4$
- This bundle has adjusted curvature (lift of doubled instanton)
- First relevant example of non-Abelian principal 2-bundle (explicit cocycles available)
- Amounts to string structure (triv. of Chern–Simons gerbe) (?)

Adjusted Connections in Supergravity

and Origins of Adjustment

Physics example: Heterotic supergravity

Archetypal example: string Lie 2-algebra

$$\mathfrak{string}(n) = \mathbb{R}[1] \to \mathfrak{spin}(n)$$

$$\mu_2(x_1, x_2) = [x_1, x_2]\ , \quad \mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3])$$

Gauge potentials: $(A, B) \in \Omega^1(U) \otimes \mathfrak{spin}(n) \oplus \Omega^2(U)$

Curvatures:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A]$$

$$H := \mathrm{d}B - \tfrac{1}{3!}(A, [A, A]) + (A, F)$$

$$= \mathrm{d}B + \underbrace{(A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])}_{\mathrm{cs}(A)}$$

Bianchi identities:

$$\mathrm{d}F + [A, F] = 0\ , \quad \mathrm{d}H - (F, F) = 0$$

SUGRA Literature (1982,1983)
Sati, Schreiber (2008)

Infinitesimal adjustment from alternators

Recall:

2-term EL_∞-algebras Roytenberg (2007)

- 2-term cochain complex $\mathfrak{E} = \mathfrak{E}_{-1} \oplus \mathfrak{E}_0$ with Leibniz bracket
- antisymmetric and Jacobi up to homotopies (alternator, ε_3).

Observation:

In example "het. supergravity", infinitesimal adjustment: alternator.

Idea: Generalize this!

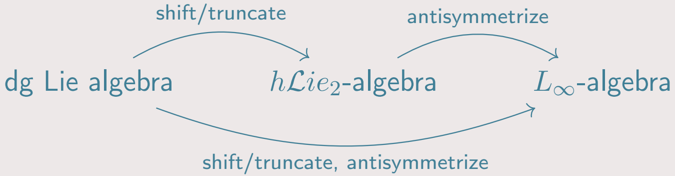
Infinitesimal adjustment from alternators

E_2L_∞-algebras — Borsten, Kim, Saemann (2021)

- homotopy algebras of $h\mathcal{L}ie_2$-algebras
- $h\mathcal{L}ie_2$-algebras:
  - cochain complex $\mathfrak{E}$
  - Leibniz-type product $\varepsilon_2^0 : \mathfrak{E} \times \mathfrak{E} \to \mathfrak{E}$,
  - alternator, controlling antisymmetry of ε_2^0.

Theorems — Borsten, Kim, Saemann (2021)

I.



II. $h\mathcal{L}ie_2$-algebras generically yield adjusted connections.

Vast generalization: Tensor hierarchies in gauged SUGRA

- Gauged supergravities come with underlying differential graded Lie algebras.
- Our formalism yields the right connections and curvatures.

Example: 5d max. supersymmetric Tensor Hierarchy

Differential graded Lie algebra (reps. of $\mathfrak{e}_{6(6)}$)

$V_{\mathfrak{e}_{6(6)}}$	$=$	V_{-5}	$\oplus$	V_{-4}	$\oplus$	V_{-3}	$\oplus$	V_{-2}	$\oplus$	V_{-1}	$\oplus$	V_0	$\oplus$	V_1
$\rho_{(k)}$		$\mathbf{27}\oplus\mathbf{1728}$		$\mathbf{351}_c$		$\mathbf{78}$		$\mathbf{27}$		$\mathbf{27}_c$		$\mathbf{78}$		$\mathbf{351}$

$h\mathcal{L}ie_2$-algebra:

$\mathfrak{E}_{\mathfrak{e}_{6(6)}}$	$=$	$\mathfrak{E}_{-4}$	$\oplus$	$\mathfrak{E}_{-3}$	$\oplus$	$\mathfrak{E}_{-2}$	$\oplus$	$\mathfrak{E}_{-1}$	$\oplus$	$\mathfrak{E}_0$
		$\mathbf{27}\oplus\mathbf{1728}$		$\mathbf{351}_c$		$\mathbf{78}$		$\mathbf{27}$		$\mathbf{27}_c$

Curvatures:

$$F^a = \mathrm{d}A^a + \tfrac{1}{2}X_{bc}{}^a A^b\wedge A^c + Z^{ab}B_b$$

$$H_a = \mathrm{d}B_a - \tfrac{1}{2}X_{ba}{}^c A^b\wedge B_c - \tfrac{1}{6}d_{abc}X_{de}{}^b A^c\wedge A^d\wedge A^e + d_{abc}A^b\wedge F^c + \Theta_a{}^\alpha C_\alpha$$

$$G_\alpha = \mathrm{d}C_\alpha - \tfrac{1}{2}X_{a\alpha}{}^\beta A^a\wedge C_\gamma + \left(\tfrac{1}{4}X_{a\alpha}{}^\beta t_{\beta b}{}^c + \tfrac{1}{3}t_{\alpha a}{}^d X_{(db)}{}^c\right)A^a\wedge A^b\wedge B_c$$
$$+\tfrac{1}{2}t_{\alpha a}{}^b F^a\wedge B_b - \tfrac{1}{2}t_{\alpha a}{}^b H_b\wedge A^a - \tfrac{1}{6}t_{\alpha a}{}^b d_{bcd}A^a\wedge A^c\wedge F^d - Y_{a\alpha}{}^\beta D_\beta{}^a$$

Geometric T-duality

A 2-group fibration

- 2-group $\mathsf{TB}_n^{\mathsf{F2}}$, string F^2-backgrounds: principal $\mathsf{TB}_n^{\mathsf{F2}}$-bundles
- There is an equivalent* 2-group TD_n:

$$\mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times \mathsf{U}(1) \rightrightarrows \mathbb{R}^{2n}$$

$$\xi \xleftarrow{(\xi, m_1, \phi_1)} \xi - m_1 \xleftarrow{(\xi - m_1, m_2, \phi_2)} \xi - m_1 - m_2$$

$$\xi \xleftarrow{(\xi, m_1 + m_2, \phi_1 + \phi_2)} \xi - m_1 - m_2$$

$$(\xi_1, m_1, \phi_1) \otimes (\xi_2, m_2, \phi_2) := (\xi_1 + \xi_2, m_1 + m_2, \phi_1 + \phi_2 - \langle \xi_1, m_2 \rangle)$$

- Double fibration of 2-groups:

$$\mathsf{TB}_n^{\mathsf{F2}} \xleftarrow{\check{\phi}} \mathsf{TD}_n \xrightarrow{\hat{\phi}} \mathsf{TB}_n^{\mathsf{F2}}$$

  - $\hat{\phi}$: strict morphism
  - $\check{\phi} = \hat{\phi} \circ \phi_{\mathsf{flip}}$ with $\phi_{\mathsf{flip}} : \mathsf{TD}_n \to \mathsf{TD}_n$

Geometric T-duality: Topological picture

2-group double fibration induces double fib. of principal 2-bundles:

$$\check{\mathscr{P}} \xleftarrow{\;\check{\mathsf{p}}\;} \mathscr{P}_{\mathrm{C}} \xrightarrow{\;\hat{\mathsf{p}}\;} \hat{\mathscr{P}}$$

- $\mathscr{P}_C$ is a principal TD_n-bundle
- $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$ are principal $\mathsf{TB}_n^{\mathsf{F2}}$-bundles
- Gerbe and circle fibration combined into 2-bundles $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
- This describes geometric ($F^2 \leftrightarrow F^2$) topological T-duality

Nikolaus, Waldorf (2018)

Geometric T-duality: Full Picture

$$\begin{array}{ccc} & \mathscr{P}_{\rm C} & \\ {\scriptstyle \check{\mathsf{p}}} \swarrow & & \searrow {\scriptstyle \hat{\mathsf{p}}} \\ \check{\mathscr{P}} & & \hat{\mathscr{P}} \end{array}$$

- Differential refinement: (i.e. B-field+metric) Kim, CS (2022)
  - TD_n comes with very natural adjustment map: $\langle -, - \rangle$
  - (interestingly, $\mathsf{TB}_n^{\mathsf{F2}}$ does not...)
  - Have topological and full connection data on $\mathscr{P}_C$
  - Can reconstruct gerbe and bundle data on $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
- Generalization to affine torus bundles: use $\mathsf{GL}(n, \mathbb{Z}) \ltimes \mathsf{TD}_n$

Verifying an example: 3d Nilmanifolds

Geometry of string background $\check{\mathscr{G}}_\ell \to N_k$:

- Principal circle bundle over T^2 with $c_1 = k$
- Subordinate to $\mathbb{R}^2 \to T^2$ and with $\mathsf{U}(1) \cong \mathbb{R}/\mathbb{Z}$

$$(x, y, z) \sim (x, y+1, z) \sim (x, y, z+1) \sim (x+1, y, z-ky)$$

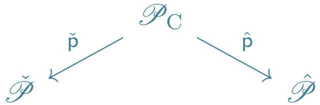
- Local connection form: $A(x, y) = kx\,\mathrm{d}y \in \Omega^1(\mathbb{R}^2)$
- Kaluza–Klein metric: $g(x, y, z) = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z + kx\,\mathrm{d}y)^2$
- Gerbes on N_k characterized by element of $H^3(N_k, \mathbb{Z}) \cong \mathbb{Z}$

T-duality:

$$(\check{\mathscr{G}}_\ell \to N_k) \;\longleftrightarrow\; (\hat{\mathscr{G}}_k \to N_\ell)$$

Explicit T-duality example with principal 2-bundles

Kim, CS (2022)



Lie 2-group:

$$\mathsf{TD}_1 \;:=\; \big(\mathbb{Z}^2 \times \mathsf{U}(1) \xrightarrow{\;\mathsf{t}\;} \mathbb{R}^2\big)$$

Topological cocycle data:

$$g = \begin{pmatrix} \hat{\xi} \\ \check{\xi} \end{pmatrix}~, \quad \begin{aligned} \hat{\xi}(x,y;x',y') &= \ell(x'-x)y~, \\ \check{\xi}(x,y;x',y') &= k(x'-x)y~, \end{aligned}$$

$$h = \begin{pmatrix} \hat{m} \\ \check{m} \\ \phi \end{pmatrix}~, \quad \begin{aligned} \hat{m}(x,y;x',y';x'',y'') &= -\ell(x''-x')(y'-y) \\ \check{m}(x,y;x',y';x'',y'') &= -k(x''-x')(y'-y) \\ \phi &= \tfrac{1}{2}k\ell\big(y'(xx''-xx'-x'x'')-(x''-x')(y'^2-y^2)x\big) \end{aligned}$$

Cocycle data of differential refinement:

$$A = \begin{pmatrix} \check{A} \\ \hat{A} \end{pmatrix} = \begin{pmatrix} kx\,\mathrm{d}y \\ \ell x\,\mathrm{d}y \end{pmatrix}~, \quad B = 0~, \quad \Lambda = \tfrac{1}{2}k\ell(xx'\,\mathrm{d}y + (xy + x'y' + y^2(x'-x))\,\mathrm{d}x)$$

Reconstruction procedure for both string backgrounds fully.

Buscher rules

Full verification:

- This formalism reproduces the Buscher rules locally.
  Waldorf (2022).

Altogether:

Full description of geometric T-duality with non-trivial topology.

The T-duality group: Automorphisms of TD_n

Abstract nonsense:

- Natural definition of morphism of 2-groups
- Automorphisms of 2-group form naturally a 2-group
- 2-group action $\mathscr{G} \curvearrowright \mathscr{H}$: morphism $\mathscr{G} \to \mathsf{Aut}(\mathscr{H})$

Automorphisms of the 2-group TD_n:

- Can be computed to be weak (unital) Lie 2-group

$$\mathscr{GO}(n,n;\mathbb{Z}) := \left(\mathsf{GO}(n,n;\mathbb{Z}) \times \mathbb{Z}^{2n} \rightrightarrows \mathsf{GO}(n,n;\mathbb{Z}) \right)$$

see also Waldorf (2022)

- While $\mathsf{GO}(n,n;\mathbb{Z})$ does not act on TD_n, $\mathscr{GO}(n,n;\mathbb{Z})$ does.
- Recover T-duality group for affine torus bundles
- Explicit: geometric subgroup, B- and β-trafos, T-dualities as endo-2-functors on TD_n
- $\Rightarrow$ arrange everything in $\mathscr{GO}(n,n;\mathbb{Z})$-covariant fashion

Non-geometric T-dualities

A proposal that requires more verification

Outline towards non-geometric spaces

So far: only geometric T-duality between F^2-backgrounds.

Recall classification of backgrounds:

F^3: H has no legs along fiber
T-duality: identity

F^2: H has 1 leg along fiber
T-duality $\rightarrow$ geometric string background

F^1: H has 2 legs along fiber
T-duality $\rightarrow$ Q-space, (e.g. T-folds) locally geometric

F^0: H has all legs along fiber
T-duality $\rightarrow$ R-space, non-geometric

Observation: T-duality is essentially a Kaluza–Klein reduction

Sati, Schreiber, Berman, Alfonsi, ...

T-duality and Kaluza–Klein reduction

Note:

- One T-duality direction: B-field $\to$ 2-, 1-forms
  $\Rightarrow$ Lie 2-group TD_n-bundles with connection
- Two T-duality directions: B-field $\to$ 2-, 1-, 0-forms
  $\Rightarrow$ Lie 2-groupoid $\mathscr{TD}_n$-bundles with connection
- Three T-duality directions: B-field $\to$ 2-, 1-, 0-, "(-1)-forms"
  (Note: (-1)-forms have global "curvature" 0-forms)

Translation to mathematics:

- 2-form B-field: abelian gerbe
- add 1-form A-field: principal 2-group bundle
- add 0-form ϕ-field: principal 2-groupoid bundle
- add -1-form ξ-field: principal augmented 2-groupoid bundle

The 2-groupoid $\mathscr{TD}_n$

- Two T-dualities yield scalars from metric and 2-form.
- Scalars live on the Narain moduli space for affine torus bundles:

$$\begin{aligned} GM_n &= \mathsf{GO}(n,n;\mathbb{Z}) \setminus \mathsf{O}(n,n;\mathbb{R}) \;/\; \big(\mathsf{O}(n;\mathbb{R}) \times \mathsf{O}(n;\mathbb{R})\big) \\ &=: \mathsf{GO}(n,n;\mathbb{Z}) \setminus Q_n \end{aligned}$$

- Note: $Q_n \cong \mathbb{R}^{n^2}$ is a nice space, "generalized metric"
- Resolve into action groupoid:

$$\mathsf{GO}(n,n;\mathbb{Z}) \ltimes Q_n \;\rightrightarrows\; Q_n$$

- Extend to $\mathscr{GO}(n,n;\mathbb{Z})$-action ($\mathscr{GO}(n,n;\mathbb{Z}) \cong \mathsf{Aut}(\mathsf{TD}_n)$)
- Place TD_n-fiber over every point in Q_n
- Extend action of $\mathscr{GO}(n,n;\mathbb{Z})$ to action on TD_n
- The result is the Lie 2-groupoid $\mathscr{TD}_n$

T-duality as $\mathscr{T}\mathscr{D}_n$-bundles

A non-geometric T-duality is simply a $\mathscr{T}\mathscr{D}_n$-bundle.

Remarks:

- The T-duality group $\mathscr{G}\mathscr{O}(n,n;\mathbb{Z}) \supset \mathsf{GO}(n,n;\mathbb{Z})$ is gauged!
- Matches topological discussion in Nikolaus, Waldorf (2018)
- This may describe all T-dualities between pairs of T-folds

To describe Q-spaces/T-folds:
(can) use higher instead of noncommutative geometry?

Example: T-fold from T^3 with H-flux

Consider again nilmanifold example with $\ell = 0$. This time $X = S^1$.

- Gauge groupoid $\mathscr{T}\mathscr{D}_2$
- Topology: all data over $Y^{[3]}$ are trivial.
- Topology: no T^n-bundles over S^1: ξ is trivial
- Remaining: $q : Y \to Q_2 \cong \mathbb{R}^4$, $g : Y^{[2]} \to \mathsf{GO}(2,2;\mathbb{Z})$ s.t.:

$$q(y_1) = g(y_1, y_2)q(y_2)\ , \quad g(y_1, y_2)g(y_2, y_3) = g(y_1, y_3)$$

- $\mathbb{R}^4$: scalar modes g_{yy}, g_{yz}, g_{zz}, B_{yz}
- Well-known T-fold is the special case where

$$g_{x+1,x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell & 1 & 0 \\ -\ell & 0 & 0 & 1 \end{pmatrix}$$

- T-duality group $\mathscr{G}\mathscr{O}(n, n; \mathbb{Z})$ acts on cocycle as expected.

What about R-spaces?

- T-folds/Q-spaces relatively harmless, as locally geometric
- R-spaces are not even locally geometric
- But perhaps higher description still works?

Note:

- One T-duality direction: B-field $\rightarrow$ 2-, 1-forms
  $\Rightarrow$ Lie 2-group TD_n-bundles with connection
- Two T-duality directions: B-field $\rightarrow$ 2-, 1-, 0-forms
  $\Rightarrow$ Lie 2-groupoid $\mathscr{T}\mathscr{D}_n$-bundles with connection
- Three T-duality directions: B-field $\rightarrow$ 2-, 1-, 0-, "(-1)-forms"
  Note: (-1)-forms have global "curvature" 0-forms
  $\Rightarrow$ Augmented Lie 2-groupoid $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$-bundles with connection

Augmented groupoid bundles

Need to switch to simplicial picture:

- (Higher) groupoids are Kan simplicial manifolds
- Higher groupoid 1-morphisms are simplicial maps
- Higher groupoid 2-morphisms are simplicial homotopies
- "quasi-groupoids" or "$(\infty, 1)$-groupoids"

Augmented $\mathscr{G}$-groupoid bundles subordinate to $\sigma : Y \twoheadrightarrow X$:

$$
\begin{array}{ccc}
Y \times_X Y \times_X Y & \xrightarrow{\;g_2\;} & \mathscr{G}_2 \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
Y \times_X Y & \xrightarrow{\;g_1\;} & \mathscr{G}_1 \\
\downarrow\downarrow & & \downarrow\downarrow \\
Y & \xrightarrow{\;g_0\;} & \mathscr{G}_0 \\
\downarrow \sigma & & \downarrow \\
X & \xrightarrow{\;g_{-1}\;} & \mathscr{G}_{-1}
\end{array}
$$

T-duality as $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$-bundles

Construction of $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$:

- Augmentation by suitable space of R-fluxes
- Determined by integrated embedding tensor of tensor hierarchy
- Beyond this, augmentation fairly trivial

Remarks on T-duality with $\mathscr{T}\mathscr{D}_n^{\mathsf{aug}}$-bundles:

- Modulis match
- All previously discussed cases included
- Yields consistency conditions between Q- and R-fluxes

To describe R-spaces:
(can) use higher instead of nonassociative geometry?

Summary

What has been done:

- (Top. T-duality can be described using principal 2-bundles)
- Differential refinement with adjusted curvatures
- Explicit description of geometric T-duality with nilmanifolds
- T-duality group is really a 2-group
- Proposal for Q-spaces or T-folds using 2-groupoid bundles
- Proposal for R-spaces using augmented 2-groupoid bundles

Future work:

- Link some mathematical results to physical expectations
  Which tests would You like to see done?
- Link to pre-NQ-manifold pictures, DFT, and similar
- Non-abelian T-duality?
- U-duality

Thank You!