# The $E_k$ Symmetry of Dimensional Reductions of M-Theory

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# Hisham Sati's "Hypothesis H" (2013)

**Hypothesis H: The dynamics of M-theory is governed by the rational homotopy theory (RHT) of $S^4$.** More precisely, the duality-symmetric equations of motion (EOMs) of supergravity in an 11d spacetime $Y$ (background of M-theory)

$$dG_4 = 0, \qquad dG_7 = -\tfrac{1}{2} G_4 \wedge G_4$$

define a map

$$\varphi : Y \to S^4$$

which induces a DGCA homomorphism

$$\varphi^* : M(S^4) \to \Omega^\bullet_{\mathrm{dR}}(Y)$$

such that

$$\varphi^*(g_4) = G_4, \quad \text{and} \quad \varphi^*(g_7) = G_7.$$

Here $\quad M(S^4) = (\mathbb{R}[g_4, g_7] \,|\, dg_4 = 0, dg_7 = -\tfrac{1}{2} g_4^2)$

is the Sullivan minimal model of $S^4$ (the RHT of $S^4$).

# More on Hypothesis H

In M-theory

$$G_4 \rightsquigarrow C_3 \rightsquigarrow \text{M2-brane}$$
$$G_7 \rightsquigarrow C_6 \rightsquigarrow \text{M5-brane}$$

Via

$$\varphi : Y \to S^4,$$

$S^4$ becomes the *universal target of M-theory* and

$$\varphi^*(\text{RHT of } S^4) = \text{EOMs of M-theory in } Y.$$

# Generalized Hypothesis H

**This pattern continues for reductions of M-theory to $d = 11 - k$ dimensions for all $k \geq 0$:**

$$\varphi_k : Y^{11}/T^k = (S^1)^k \to \mathcal{L}_c^k S^4$$

**with $\mathcal{L}_c^k S^4$ serving as a universal target space for $(11 - k)$-dim M-theory!!!**

Here

$$\mathcal{L}_c^k Z := \mathcal{L}_c(\mathcal{L}_c(\ldots(\mathcal{L}_c Z)\ldots)),$$
$$\mathcal{L}_c Z := \mathcal{L}Z /\!\!/ S^1 := \mathcal{L}Z \times_{S^1} ES^1,$$
$$\mathcal{L}Z := \mathrm{Map}(S^1, Z),$$

is the *iterated cyclic loop space* (*cyclification*) $\mathcal{L}_c^k Z$ of $Z$.

# Generalized Hypothesis H

$$
\begin{array}{lccc}
\text{11d M-theory spacetime} & Y^{11} & \xrightarrow{\varphi} & S^4 \\
& \downarrow & & \\
\text{Type IIA string in 10d} & Y^{11}/S^1 & \xrightarrow{\varphi_1} & \mathcal{L}_c S^4 \\
& \downarrow & & \\
\text{9d supergravity spacetime} & Y^{11}/(S^1)^2 & \xrightarrow{\varphi_2} & \mathcal{L}_c^2 S^4 \\
& \downarrow & & \\
& \cdots & &
\end{array}
$$

with all the EOMs on $X^{11-k} = Y^{11}/(S^1)^k$ may be read off from the Sullivan minimal model $M(\mathcal{L}_c^k S^4)$ of $\mathcal{L}_c^k S^4$, which is well-known, see next slides.

# Example: Type IIA string theory

(as per Fiorenza-Sati-Schreiber 2017)

$$X^{10} = Y^{11} /\!\!/ S^1 \xrightarrow{\varphi_1} \mathcal{L}_c S^4$$

$$\begin{gathered} M(\mathcal{L}_c S^4) = (\mathbb{R}[g_4, g_7, sg_4, sg_7, w], d), \\ |w| = 2, \qquad |sg_4| = 3, \qquad |sg_7| = 6, \\ dg_4 = (sg_4) \cdot w, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + (sg_7) \cdot w, \\ d(sg_4) = 0, \quad d(sg_7) = (sg_4) \cdot g_4, \quad dw = 0. \end{gathered}$$

In standard physics notation:

$$F_2 := \varphi_1^*(w), \quad H_3 := \varphi_1^*(sg_4), \quad F_4 := \varphi_1^*(g_4), \quad H_7 := \varphi_1^*(g_7).$$

Equations of motion (EOMs) of 10d type-IIA supergravity:

$$\begin{gathered} dF_4 = H_3 \wedge F_2, \qquad dH_7 = -\tfrac{1}{2} F_4 \wedge F_4 + F_6 \wedge F_2, \\ dH_3 = 0, \qquad dF_6 = H_3 \wedge F_4, \qquad dF_2 = 0. \end{gathered}$$

# Hypothesis H $\Rightarrow$ Principle H:

**Principle H**

*Any feature of or statement about the Sullivan minimal model $M(\mathcal{L}_c^k S^4)$ of an iterated cyclic loop space $\mathcal{L}_c^k S^4$ (or the rational homotopy type thereof) may be translated into a feature of or statement about the compactification of M-theory on the $k$-torus $T^k = (S^1)^k$.*

**Preparation**: There are adjunctions
(1) In **Topology**:

$$\mathrm{Map}(E(Y), Z) \xrightarrow{\sim} \mathrm{Map}_{/BS^1}(Y, \mathcal{L}_c Z),$$

where $E(Y)$ is the total space of the principal $S^1$-bundle over $Y$ corresponding to $Y \to BS^1$, see Braunack-Mayer, Sati, and Schreiber (2018). This property determines $\mathcal{L}_c Z$, up to unique homeomorphism over $BS^1$.
(2) In **Algebra**:

$$\mathrm{Hom}_{\mathbb{R}\text{-DGCA}}(M, N/(w)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}[w]\text{-DGCA}}(\mathrm{VPB}(M), N),$$

where $\mathrm{VPB}(M)$ is the “Vigué-Poirrier-Burgheleazation” of $M$, see Vigué-Poirrier and Burghelea (1985). This property determines $\mathrm{VPB}(M)$, up to unique isomorphism of dg-$\mathbb{R}[w]$-algebras.

**Translation**: $M = M(Z)$, $N = M(Y)$, $\mathbb{R}[w] = M(BS^1)$, $N/(w) = M(E(Y))$, $\mathrm{VPB}(M) = M(\mathcal{L}_c Z)$.

# The Unreasonable Effectiveness of Hypothesis H

(1) $$\mathrm{Map}(E(Y), Z) \xrightarrow{\sim} \mathrm{Map}_{/BS^1}(Y, \mathcal{L}_c Z)$$

(2) $$\mathrm{Hom}_{\mathbb{R}\text{-DGCA}}(M, N/(w)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}[w]\text{-DGCA}}(\mathrm{VPB}(M), N)$$

(3) In **M-Theory**:

$$\{\text{M-theory of type } M \text{ on } E(Y)\} \xrightarrow{\sim} \{\text{M-theory of type } R(M) \text{ on } Y\},$$

where the type of M-theory is expected to be determined by the dimension of the spacetime and we are actually speaking about pre-M-theories or **duality-symmetric supergravities**, to be precise. What constitutes a dimensionally reduced M-theory is exactly the "Vigué-Poirrier–Burgheleazation" of M-theory before the reduction. This property determines what the reduction is, up to unique change of variables for fields.

**Translation**: A reduced M-theory of Type $M$ on spacetime $Y$ = a DGCA homomorphism $M \to \Omega^\bullet_{\mathrm{dR}}(XY$ for a certain DGCA $M$.

**Consequence**: An M-theory on $X^{11-k}$ = a DGCA homomorphism $M(\mathcal{L}_c^k S^4) \to \Omega^\bullet_{\mathrm{dR}}(X)$.

# Recipe for EOMs of Dimensional Reduction

## Vigué-Poirrier-Burgheleazation of Supergravity

If supergravity on a spacetime $X$ is given by form fields $G_1$, $G_2$, $\ldots$, with EOMs

$$dG_1 = p_1(G_1, G_2, \ldots), \quad dG_2 = p_2(G_1, G_2, \ldots), \ldots,$$

then its reduction to $X/S^1$ will be given by form fields

$$G_1, G_2, \ldots, \quad SG_1, SG_2, \ldots, \quad \omega,$$

$\deg SG_i = \deg G_i - 1$, $\deg \omega = 2$, with EOMs

$$dG_i = p_i(G_1, G_2, \ldots) + SG_i \wedge \omega,$$
$$dSG_i = -Sp_i(G_1, G_2, \ldots), \quad d\omega = 0,$$

where $S$ is extended as a square-zero derivation of the algebra of polynomials $\mathbb{R}[G_1, G_2, \ldots]$.

# Toroidification versus Iterated Cyclification

**Use toroidication $\mathcal{T}^k S^4 := \mathcal{L}^k S^4 /\!\!/ T^k$ instead of iterated cyclification $\mathcal{L}_c^k S^4$ to allow for more symmetry**

| **Cyclification** | **Toroidification** |
|---|---|
| $\mathcal{L}_c^k S^4 = \mathcal{L}(\ldots(\mathcal{L}S^4 /\!\!/ S^1)\ldots)/\!\!/ S^1$ | $\mathcal{T}^k S^4 = \mathcal{L}^k S^4 /\!\!/ T^k$ |
| More axions, like $S_2\omega_1$ | Some axions, like $S_3 S_2 S_1 G_4$ |
| Same toroidal symmetry | Same toroidal symmetry |
| Less non-abelian symmetry | More non-abelian symmetry |
| $M(\mathcal{L}_c^k S^4) = \mathrm{VPB}^k(M(S^4))$ | $M(\mathcal{T}^k S^4) = \mathrm{SV}_k(M(S^4))$ |

**Defined by different adjunctions**:

$$\mathrm{Hom}_{\mathbb{R}\text{-DGCA}}(M, N/(w)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}[w]\text{-DGCA}}(\mathrm{VPB}(M), N)$$

$$\mathrm{Hom}_{\mathbb{R}\text{-DGCA}}(M, N/(w_1, \ldots, w_k)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}[w_1,\ldots,w_k]\text{-DGCA}}(\mathrm{SV}_k(M), N)$$

# Example of a Toroidification

$$M(\mathcal{T}^3 S^4) = \left(\mathbb{R}[g_4, g_7, s_i g_4, s_i g_7, s_i s_j g_4, s_i s_j g_7, s_i s_j s_k g_4, s_i s_j s_k g_7, w_i], d\right),$$

$$1 \leq i, j, k \leq 3, \qquad s_j s_i = -s_i s_j,$$

$$dg_4 = \sum_{i=1}^{3} s_i g_4 \cdot w_i, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + \sum_{i=1}^{3} s_i g_7 \cdot w_i,$$

$$ds_i g_4 = \sum_{j=1}^{3} s_j s_i g_4 \cdot w_j, \quad ds_i g_7 = s_i g_4 \cdot g_4 + \sum_{j=1}^{3} s_j s_i g_7 \cdot w_j,$$

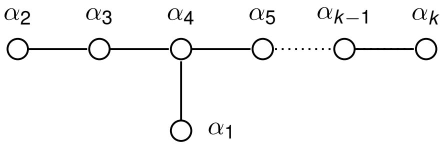
$$ds_i s_j g_4 = s_i s_j s_k g_4 \cdot w_k, \quad k \neq i, j,$$

$$ds_i s_j g_7 = -s_i s_j g_4 \cdot g_4 - s_i g_4 \cdot s_j g_4 + s_i s_j s_k g_7 \cdot w_k, \quad k \neq i, j$$

$$ds_1 s_2 s_3 g_4 = 0, \quad ds_1 s_2 s_3 g_7 = s_1 s_2 s_3 g_4 \cdot g_4 + \sum_{\substack{i<j \\ k \neq i,j}} \operatorname{sgn}\left(\begin{smallmatrix} 1 & 2 & 3 \\ i & j & k \end{smallmatrix}\right) s_i s_j g_4 \cdot s_k g_4,$$

$$dw_i = 0, \qquad 1 \leq i \leq 3.$$

# The $E_k$ series




The Cartan matrix

$$C = (c_{ij}) = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

is almost the negative of the incidence matrix of the Dynkin diagram (except for the diagonal entries $c_{ii} = 2$).

# More on the $E_k$ series

| $k$ | Universal Target | Type of $E_k$ | Lie Algebra $\mathfrak{e}_k$ |
|---|---|---|---|
| 0 | $S^4$ | $A_{-1}$ | $\mathfrak{sl}_0 = \varnothing$ |
| 1 | $\mathcal{T}^1 S^4$ | $A_0$ | $\mathfrak{sl}_1 = 0$ |
| 1 | IIB | $A_1$ | $\mathfrak{sl}_2$ |
| 2 | $\mathcal{T}^2 S^4$ | $A_1$ | $\mathfrak{sl}_2$ |
| 3 | $\mathcal{T}^3 S^4$ | $A_2 \times A_1$ | $\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$ |
| 4 | $\mathcal{T}^4 S^4$ | $A_4$ | $\mathfrak{sl}_5$ |
| 5 | $\mathcal{T}^5 S^4$ | $D_5$ | $\mathfrak{so}_{10}$ |
| 6 | $\mathcal{T}^6 S^4$ | $E_6$ | $\mathfrak{e}_6$ |
| 7 | $\mathcal{T}^7 S^4$ | $E_7$ | $\mathfrak{e}_7$ |
| 8 | $\mathcal{T}^8 S^4$ | $E_8$ | $\mathfrak{e}_8$ |
| 9 | $\mathcal{T}^9 S^4$ | $E_9$ | affine $\mathfrak{e}_9 = \widehat{\mathfrak{e}_8}$ |
| 10 | $\mathcal{T}^{10} S^4$ | $E_{10}$ | hyperbolic $\mathfrak{e}_{10}$ |

# Lie algebra of type $E_k$

We will define a (trivial) central extension $\mathfrak{g}_k$ of the split real form $E_{k(k)}$ of the Lie algebra $\mathfrak{e}_k$ of type $E_k$ (assume $k \geq 3$).

Fix a real vector space $\mathfrak{h}_k$ of dimension $k+1$. We assume a set of $k$ linearly independent elements (thought of as *simple coroots*) $\alpha_i^\vee$ of $\mathfrak{h}_k$ and a set of $k$ linearly independent elements (thought of as *simple roots*) $\alpha_i$ of the dual space $\mathfrak{h}_k^*$, such that $\alpha_i(\alpha_j^\vee) = c_{ji}$, are given.

The Lie algebra $\mathfrak{g}_k$ is defined by (*Chevalley*) generators $e_1, e_2, \dots, e_k, f_1, f_2, \dots, f_k$ and the elements of $\mathfrak{h}_k$ and relations:

- $[h, h'] = 0$ for $h, h' \in \mathfrak{h}_k$;
- $[h, e_i] = \alpha_i(h) e_i$ for $h \in \mathfrak{h}_k$;
- $[h, f_i] = -\alpha_i(h) f_i$ for $h \in \mathfrak{h}_k$;
- $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$;
- If $i \neq j$ (so $c_{ij} \leq 0$) then $\mathrm{ad}(e_i)^{1-c_{ij}}(e_j) = 0$ and $\mathrm{ad}(f_i)^{1-c_{ij}}(f_j) = 0$ (*Serre relations*).

We consider the **parabolic subalgebra** $\mathfrak{p}_k$ generated by all the generators but $e_1$ and define an action of $\mathfrak{p}_k$ on $M(\mathcal{T}^k S^4)$, as follows:

$$f_1 g_4 = f_1 g_7 = f_1 s_i g_4 = f_1 s_i g_7 = f_1 s_i s_j g_4$$
$$= f_1 s_i s_j g_7 = f_1 s_i s_j s_l g_4 = f_1 s_i s_j s_l g_7 = f_1 w_i = 0,$$
$$\text{except for } f_1 s_i s_j g_4 = \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 \\ i & j & l \end{pmatrix} w_l \quad \text{for } \{i, j, l\} = \{1, 2, 3\}$$
$$\text{and } f_1 s_1 s_2 s_3 g_7 = g_4,$$
$$[f_1, s_i] = 0, \qquad 4 \le i \le k,$$
$$h g_4 = \varepsilon_0(h) g_4, \quad h g_7 = 2\varepsilon_0(h) g_7,$$
$$[h, s_i] = -\varepsilon_i(h) s_i, \quad h w_i = \varepsilon_i(h) w_i, \qquad 1 \le i \le k,$$
$$e_i g_4 = e_i g_7 = 0, \qquad 2 \le i \le k,$$
$$e_i w_j = \delta_{ij} w_{i-1}, \qquad 2 \le i \le k, 1 \le j \le k,$$
$$[e_i, s_j] = -\delta_{i-1,j} s_i, \qquad 2 \le i \le k, 1 \le j \le k,$$
$$f_i g_4 = f_i g_7 = 0, \qquad 2 \le i \le k,$$
$$f_i w_j = \delta_{i-1,j} w_i, \qquad 2 \le i \le k, 1 \le j \le k,$$
$$[f_i, s_j] = -\delta_{ij} s_{i-1}, \qquad 2 \le i \le k, 1 \le j \le k.$$

# Main Theorems: U-Duality in $(11 - k)$-Dim M-Theory

## Theorem 1

The above formulas define a (linear) action of the **parabolic** Lie subalgebra $\mathfrak{p}_k$ of $\mathfrak{g}_k$ on the Sullivan minimal model $M(\mathcal{T}^k S^4)$, *i.e.*, a Lie-algebra homomorphism

$$\mathfrak{p}_k \to \operatorname{Der} M(\mathcal{T}^k S^4).$$

## Theorem 2

The action of $\mathfrak{p}_k$ on $M(\mathcal{T}^k S^4)$ may be extended to an action of $\mathfrak{g}_k$ by (linear) graded derivations in $\operatorname{Der}^0 M(\mathcal{T}^k S^4)$. In other words, we have a commutative diagram of Lie-algebra homomorphisms

$$\begin{array}{ccc} \mathfrak{p}_k & \longrightarrow & \operatorname{Der} M(\mathcal{T}^k S^4) \\ \downarrow & & \downarrow \\ \mathfrak{g}_k & \longrightarrow & \operatorname{Der}^0 M(\mathcal{T}^k S^4). \end{array}$$

# Meaning of Theorem 2

The component $L^0$ of a dg-Lie algebra $L^\bullet$ plays the role of a *gauge Lie algebra* in deformation theory governed by $L^\bullet$. In the case of a Sullivan minimal model $M(X)$, the dg-Lie algebra $\mathrm{Der}^\bullet M(X)$ describes the deformation theory of the Quillen minimal model $Q(X)$ as an $L_\infty$-algebra. In the case of $X = \mathcal{T}^k S^4$, the Quillen minimal model is just a graded Lie algebra, the rational homotopy Lie algebra $\pi_\bullet(\mathcal{T}^k S^4)[1] \otimes \mathbb{Q}$ of $\mathcal{T}^k S^4$ with respect to the Whitehead product. From the physics perspective, the **Lie algebra $\mathfrak{g}_k$ acts by gauge transformations in the $L_\infty$ deformation theory of the gauge Lie algebra $\pi_\bullet(\mathcal{T}^k S^4)[1] \otimes \mathbb{Q}$ of the reduction of M-theory on the torus $T^k$**. This gauge Lie algebra is a $k$-fold dimensional reduction of the M-theory gauge algebra $\pi_\bullet(S^4)[1] \otimes \mathbb{Q}$, which is based on generators $x_3$ of degree 3 and $x_6$ of degree 6:

$$[x_3, x_3] = x_6, \qquad [x_3, x_6] = 0.$$