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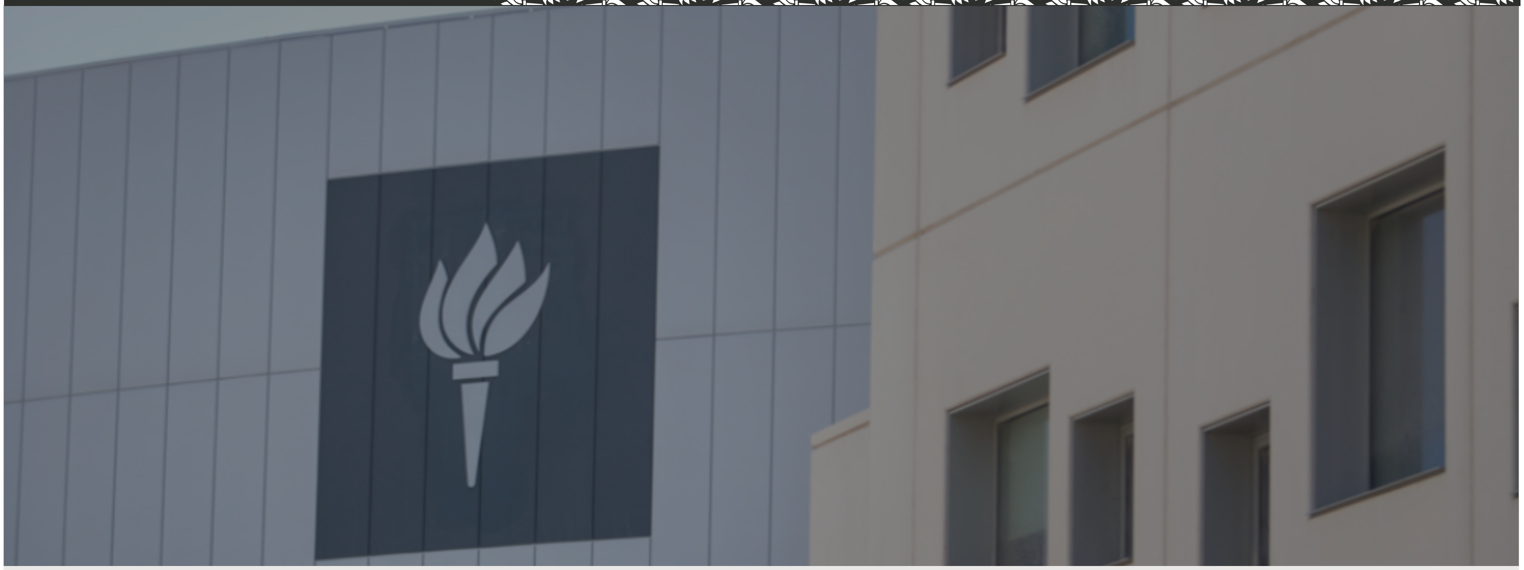

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Bounding Comparative Statics under Diagonal Dominance

J.J. Norris, C.R. Johnson, I.M. Spitkovsky*

January, 2023

Abstract

A core purpose of economic modeling is to conduct comparative static analyzes. Often one is interested in its qualitative features, such as if the effect of a shock is positive or above one. Yet, except in highly-stylized models, the theoretically implied relationships are intractable, and empirically demanding, requiring complete identification of the model. We derive new bounds on comparative statics that are more tractable and feasible under partial identification. We require only that the Jacobian is diagonally dominant — intuitively, there is limited feedback in the model. We demonstrate application in two canonical models: a network game and a model of oligopoly competition.

Keywords: comparative statics, diagonal dominance, networks, qualitative economics
JEL Classification: C3, D85

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1 Introduction

A core purpose of modeling in economics is to conduct comparative statics: the analysis of the effect of an exogenous shock on an endogenous outcome. Except in highly stylized models (an assumption becoming increasingly untenable in the age of big data), the equations linking these variables are very complicated. Even when linearized in the outcome and shock, the equations are generally highly non-linear in the other parameters of the model. Moreover, even when the the shock and outcome of interest are confined to single node (a single agent, location, market, etc), the comparative static depends on the properties, and responses, of all other nodes in the model.

At least two important challenges arise from these features. First, being highly non-linear, the qualitative properties of the comparative statics are hard to prove — such as sufficient conditions for the effect to be positive or bounded below by some number.[1] Second, unless strong symmetry assumptions are imposed (such as assuming the nodes are linked in a pairwise symmetric manner), quantifying a comparative static in the model is very demanding empirically. When linearized, with N nodes there are N^2 interactions, all possibly distinct, and all possibly unobservable. An simple example demonstrating this that we consider is an oligopoly game with N differentiated firms and linear demand (Pellegrino (2019); Galeotti et al. (2022)). The effect of a firm's cost shock to its own profits depends on N^2 distinct (and empirically unobservable) bilateral elasticities of substitution in an infinite-order multivariate polynomial.

In this paper, we provide new results that allow progress on these challenges. We derive new bounds on (linearized) comparative statics that, relative to the exact relationship, have a simpler functional dependence on, and do not require as complete knowledge of, the underlying parameters of the model. This thus reduces the burden of each of the aforementioned challenges. The trade-off of is that the comparative static is only partially identified — a bound — and therefore its usefulness depends on the research question.

The comparative statics bounds we derive are valid in a wide range of models. For our main results, the only substantive assumption imposed is that the Jacobian of the model is diagonally dominant. This intuitively can be understood as implying that the endogenous feedback in the system is greater within a node relative to between nodes (Arrow and Hahn (1971) p.242). Diagonal dominance has a long history in economics in the study of comparative statics McKenzie (1960); Hadar (1965); Fujimoto (1987); Christensen (2019), and is often invoked in models to guarantee uniqueness or stability of equilibria Gale and Nikaido

[1] Qualitative properties have had historically large appeal in the literature Bassett et al. (1967); Milgrom and Shannon (1994); Athey et al. (1998); Hale et al. (1999). This is because they can be robust to, or agnostic about, specific (quantitative) model assumptions.

(1965); Dixit (1986); Adão et al. (2019); Allen et al. (2022); Arrow and Hahn (1971) T.9.12 (p. 234).[2] Relative to the literature, we take the same starting point (diagonal dominance) but derive new implications (comparative statics bounds). The literature has only found diagonal dominance to be helpful in determining the sign of a comparative static; we extend this to show it can also be useful in bounding the magnitude.

In deriving these results, we make a technical contribution to the linear algebra literature. Formally, we show that for any matrix that is diagonally dominant, the principal minors are greater than the off-diagonal minors. Moreover, we show they are greater by a factor that describes the degree of diagonal dominance in the matrix. This result can be understood as a generalization of M-matrix theory Johnson (1982). We then show that when this result is applied to comparative statics, with the Jacobian assumed to be diagonally dominant, one is able to derive bounds on the comparative static expressions.

The simplification achieved by the comparative static bounds can be understood as follows. The comparative static describes the effect of a shock on a node to an outcome on that or some other node (e.g. the effect of a firm's productivity shock on another firm's price). In general, the outcomes of all nodes respond to this shock due to feedback in the system (e.g. a firm changes its price in response to other firms' changing their prices). The comparative static captures the total effect of this feedback reverberation throughout the system. The bilateral feedback between each pair of nodes is described mathematically by the Jacobian matrix, and it turns out the total reverberation is appropriately represented by the inverse of this Jacobian matrix (a Leontieff Inverse under certain assumptions, as in Carvalho and Tahbaz-Salehi (2019)). Except when of size 2×2, inverse matrices are very complicated objects: they are a highly non-linear aggregation of the elements in the (non-inverted) Jacobian. In economics models, these matrices are massive under realistic heterogeneity (e.g. equal to the number of firms in a marketplace).

The power of the comparative static bound derived in this paper is that it is a function of the *non-inverted* Jacobian matrix only. There are two main implications of this that we highlight. First, it generally has a much simpler dependence on the underlying parameters of the model, relative to the exact expression for the comparative static. And, second, for a comparative static between a single pair of nodes, it doesn't require knowledge of all elements of the Jacobian matrix, whereas the inverse of the Jacobian, and therefore exact expression for the comparative static, does.

To understand how a bound is possible without inverting the matrix, consider the special case where there is no feedback between nodes. The comparative static requires no matrix

[2] In Allen et al. (2022), this is implied in the case of a single type $H = 1$ under the generalized domain and range (remark 1).

inverse as there is no reverberation (e.g. a model with a single firm). Relative to this no feedback-between-nodes case, the comparative static may be bigger or smaller depending on whether the feedback configuration causes amplification or attenuation. The bound we derive delivers the most conservative case: maximal attenuation. It turns out, under diagonal dominance of the Jacobian, this case is calculable without inverting the Jacobian. We show using simulations that the bound is tighter when there is less overall feedback in the system — as this intuition would suggest.

We end the paper by presenting two examples to demonstrate application of these bounds to comparative statics. A model of a network game, and an oligopoly pricing model — as referenced above. In both we use standard assumptions from the literature. To highlight just one result our tools imply. In the oligopoly model, we find that the effect of cost shock to a firm on its quantity of production can be bounded above by (one half of) the inverse of the (inverse) own-price effect of demand. This is a massive simplification relative to the exact expression, which depends on all bilateral price effects between firms, in an infinite polynomial.

The outline of this paper is as follows. In section 2, we outline the general framework. In section 3, we present our new linear algebraic results on diagonal dominance. In section 4, we present the general implications of these for comparative statics. In section 5, we demonstrate these results in various concrete models. In section 6, we conclude.

2 Model

Consider a system of $i \in \{1, ..., N\} \equiv \mathcal{N}$ nodes. The nodes may represent goods, countries, agents, etc. Each node has an endogenous state $y_i \in \mathbb{R}$ (e.g. the price of good i) and is subject to an exogenous shock $x_i \in \mathbb{R}$ (e.g. the productivity in producing good i). The state of all nodes is determined jointly by the following equations of state

$$\forall i \in \{1, ..., N\}: \quad 0 = f_i(\mathbf{y}, \mathbf{x}) \tag{1}$$

where $\mathbf{y} = \{y_j\}_{j=1}^N, \mathbf{x} = \{x_j\}_{j=1}^N$. The function $f_i : \mathbb{R}^{2N} \to \mathbb{R}$ is assumed differentiable, and is typically derived from the equilibrium conditions in the underlying economic model. For a given vector of exogenous shocks, $\mathbf{x}$, we denote a solution to equation (1) by $\mathbf{y}^*$, where the dependence on $\mathbf{x}$ is suppressed. In general the solution needn't be unique. We refer to the Jacobian matrix, ∇_{ij} of this function as the partial derivate of f_i with respect to the endogenous variable y_i, that is

$$\nabla_{ij} = \frac{\partial f_i(\mathbf{y}, \mathbf{x})}{\partial y_j}$$

where the dependence of ∇ on $\{\mathbf{y}, \mathbf{x}\}$ is suppressed.

The key substantive assumption we make is that at a solution $\mathbf{y}^*$, the Jacobian matrix is diagonally dominant – assumption 1.[3]

Assumption 1. (Diagonal Dominance). *At $\mathbf{y}^*$, the Jacobian is strictly column diagonally dominant,*

$$\forall i \in \{1, ..., N\}: \quad |\nabla_{ii}| > \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$$

$|x|$ is the absolute value of some $x \in \mathbb{R}$. Although this is potentially restrictive, this assumption has a long history in economics McKenzie (1960). In particular, in the characterization of qualitative properties of comparative statics and in stability analyses (Hadar (1965), Bassett et al. (1967), Dixit (1986), Hale et al. (1999)). It is well-known that assumption 1 implies invertibility of the Jacobian, which therefore guarantees local uniqueness (Mas-Colell et al. (1995), proposition 17.D.1). Moreover, if one further assumes that the diagonal elements are all positive ($\forall i : \nabla_{ii} > 0$), which combined with diagonal dominance implies the Jacobian is a P-matrix, then global uniqueness is guaranteed within a closed rectangular region for which both of these assumptions hold (Gale and Nikaido (1965) theorem 4).[4] The diagonal being positive is often implied by invoking the equilibrium to be stable (Hahn (1982) theorem T.1.7).

Comparative statics of the system are considered by taking an infinitesimal perturbation in $\mathbf{x}$ about a solution $\mathbf{y}^*$. We only consider consider first order effects in this paper. Applying the implicit function theorem to equation (1), we solve for the infinitesimal change in the state as

$$\frac{\partial y_i}{\partial x_j} = -\sum_{k \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \nabla^x_{kj} \tag{2}$$

where $\nabla^x_{kj} = \frac{\partial f_k(\mathbf{y},\mathbf{x})}{\partial x_j}$ we refer to as the direct effect matrix.[5] Note in particular that the comparative static $\frac{\partial y_i}{\partial x_j}$ depends on the inverse of Jacobian. Intuitively, the matrix inverse shows up in the comparative static as it appropriately aggregates all the endogenous feedback in the system (a Leontieff Inverse under certain assumptions, as in Carvalho and Tahbaz-Salehi (2019)). That is, the effect of a shock in j on outcome in i incorporates not only the

[3] Analogous results follow if one instead assumes *row* diagonal dominance. Though the results are more limited in the case of non-local direct effects.

[4] Intuition for this result can be understood as follows. Multiple solutions to equation (1) can occur when the gradient of function f_i — the Jacobian — "changes sign", thus allowing the function to cross zero at multiple points. Diagonal dominance of the Jacobian essentially implies we can ignore the cross derivatives in determining the sign. A positive diagonal of the Jacobian implies f_i always moves in the same direction, and thus does not change sign (whereas a negative diagonal means f_i reverses direction and thus does change sign). Hence the combination of these two is sufficient for a single crossing of f_i at zero, and and thus a unique solution.

[5] When y represents the price, ∇^x is referred to as the price effect matrix Mas-Colell et al. (1995).

direct effect of x_j on y_i, but also the indirect effect via the changes in states of all other nodes. States in other nodes $y_{j\neq i}$ respond to a change in state y_i, and this in turn causes y_i to change again. This feedback between nodes is precisely what the Jacobian describes.

Matrix inverses are generally very complicated (except under high symmetry or $N = 2$). Moreover, even when one is interested in identifying the comparative static only between a single pair of nodes, the inverse implies that complete identification of the entire $N \times N$ Jacobian matrix is required. This makes the analysis of comparative statics challenging: the former in determining its qualitative properties, and the latter in the empirical burden required for identification.

In the next section, we derive new results that allow one to place bounds on the comparative static under assumption 1. The key feature of these bounds is that they do not depend on the inverse of the Jacobian, instead only in its non-inverted form. Thus, potentially alleviating the two issues discussed above.

3 New Properties of Diagonally Dominant Matrices

In this section we present our new results on diagonal dominance. These results apply generally, without the matrix needing to have the interpretation as a Jacobian. In order to present these, we first need to define an object that describes the (inverse) intensity of the diagonal dominance in the matrix, which we refer to as its "degree", as it is a generalization of the degree centrality (number of other nodes a node is connected to) of a node in a network model.[6] For any matrix ∇, define the degree of node i by

$$\delta_i \equiv \frac{\sum_{j\neq i} |\nabla_{ji}|}{|\nabla_{ii}|} \tag{3}$$

where $\delta_i \in [0, 1)$ under diagonal dominance of ∇ (assumption 1). Denote the maximal degree by

$$\delta^* \equiv \max_i \delta_i \tag{4}$$

When $\delta^* = 0$, then ∇ is a diagonal matrix and therefore maximally diagonally dominant. As we increase any δ_i up from 0, the intensity of diagonal dominance diminishes.

Under the model in section 2, δ_i can be understood as describing the magnitude of feedback between nodes in the system. If $\delta^* = 0$, then the equation of state for some i, f_i, in equation (1) depends only on its own state y_i. This is true for all equations of state,

[6] In section 5.1, we show that δ_i is proportional to the standard concept of degree in a canonical network model. See equation (18).

thus each y_i is determined independently of one another, thus shutting down all endogenous feedback between the nodes. As δ_i increases, the feedback between nodes increases.

Armed with this definition, we are now in a position to present the key novel linear algebraic result. Let $\nabla_{-i,-j}$ denote the (i, j) sub-matrix of ∇ (row i and column j removed), and $\det \nabla$ the determinant of Ω.

Theorem 1. (Minors of Diagonally Dominant Matrices). *Suppose that ∇ is column diagonally dominant, then*

$$\forall j, i \neq j : \quad |\det \nabla_{-j,-i}| < \delta^* |\det \nabla_{-i,-i}| \tag{5}$$

where δ^* indicates the maximal degree of ∇ — equation (4).

Proof: see appendix A.1.

In words, for any diagonally dominant matrix, the principal minors of the matrix, $|\det \nabla_{-i,-i}|$, are greater than the off-diagonal minors within the same column, $|\det \nabla_{-j,-i}|$.[7] Moreover, they are greater by a factor $1/\delta^* \geq 1$.

This proposition has a nice intuitive appeal because the transpose of the matrix of minors is proportional to the inverse of the original matrix (in particular, $\{\nabla^{-1}\}_{ij} = \det \nabla_{-j,-i}/ |\det \nabla|$). Thus, if the original matrix is *diagonally dominant*, the theorem tells us the inverted matrix satisfies a dual property: its *diagonal elements are dominant of the (row) off-diagonal elements*. Note this condition on the inverse is weaker, as it is not sufficient for the inverse to be diagonally dominant: $|\{\nabla^{-1}\}_{ii}|$ is greater than $\left|\{\nabla^{-1}\}_{ij}\right|$, but not necessarily greater than $\sum_{j\neq i} \left|\{\nabla^{-1}\}_{ij}\right|$.

Theorem 1 has interest in its own right, removed from its consequences for comparative statics that we focus on. It can be understood as a significant extension of the well-known M-matrix theory (Johnson (1982)). M-matrices are diagonally dominant with positive diagonal elements, and negative off-diagonal elements (in terms of the model in section 2: only positive feedback is permitted). Theorem 1 applies to matrices with elements of any sign, therefore applies to a larger class of matrices.

Moreover, theorem 1 provides novel properties even for the class of M-matrices. It is well-known that inverses of M-matrices have diagonal elements that are dominant of the off-diagonal elements, but by the factor $1/\delta^*$ given in theorem 1 is hitherto not known.

Theorem 1 implies two important novel properties, as listed in corollary 1.

Corollary 1. (Properties of Diagonally Dominant Matrices). *Suppose that ∇ is column diagonally dominant, then,*

[7] A minor is the determinant of a sub-matrix, with a principal minor being of the sub-matrix along the diagonal.

i)

$$\forall i, j \neq i: \quad \left| \left\{ \nabla^{-1} \right\}_{ij} \right| < \delta^* \left| \left\{ \nabla^{-1} \right\}_{ii} \right| \tag{6}$$

ii)

$$\forall i: \quad \left| \left\{ \nabla^{-1} \right\}_{ii} \right| > \frac{1}{|\nabla_{ii}|} \cdot \frac{1}{1 + \delta_i \delta^*} \tag{7}$$

where δ_i and δ^* indicate the degree of ∇ — equations (3) and (4).

Proof*: Part i) follows from the cofactor form of the inverse, $\left\{ \nabla^{-1} \right\}_{ij} = \det \nabla_{-j,-i} / |\det \nabla|$, and theorem 1. Part ii) follows from*

$$|\det \nabla| = \left| \sum_{j=1}^{N} (-1)^{i+j} \nabla_{ij} det \left(\nabla_{-i,-j} \right) \right| \leq \sum_{j=1}^{N} |\nabla_{ij}| \, \delta^* \, |\det \nabla_{-i,-i}| \leq |\nabla_{ii}| \left(1 + \delta_i \delta^* \right) |\det \nabla_{-i,-i}|$$

where the first equality uses the Laplace formula for the determinant; the second (in)equality follows from theorem 1; the third (in)equality uses the definition of degree, equation (3). Equation (7) follows from this inequality and the cofactor form of the inverse, $\left\{ \nabla^{-1} \right\}_{ii} = \det \nabla_{-i,-i} / |\det \nabla|$.

Part i) follows immediately from equation (5) as the inverse is proportional to the transposed matrix of minors.[8] Part ii) follows from using the Laplace formula to write the determinant $|\det \nabla|$ in terms of its minors. We can use theorem 1 to substitute out the non-principal minors in this expression, leaving an upper bound on $|\det \nabla|$ in terms of only the principal minor $|\det \nabla_{-i,-i}|$. Using the cofactor form of the inverse, $\left\{ \nabla^{-1} \right\}_{ii} = \det \nabla_{-i,-i} / |\det \nabla|$, we get a bound on the diagonal of the inverse.

4 Implications for Comparative Statics

We now apply the results of section 3 in order to derive bounds on the comparative statics, equation (2). In order to present these results with the most clarity, we first detail the implications under the simplification that $\nabla^x_{i,j \neq i} = 0$, which we refer to as local direct effects only, and after go on to discuss our findings in the more general case.

4.1 Local Direct Effects Only

Assuming direct effects are local, i.e the direct effect matrix is a diagonal matrix $\forall i, j \neq i : \nabla^x_{ij} = 0$, implies that a exogenous shock in node i, x_i, only directly effects equation of

[8] Using a different method of proof, Fujimoto (1987) proved a weaker version of corollary 1.i): equation (6) with δ^* set to one. This also implies weaker versions — all with δ^* set to one — of parts ii) of corollary 1, and of theorem 5. Though none of these are present in his paper.

state i.[9] Of course, there is still an indirect effect on other equations of state through the endogenous feedback (which is described by the Jacobian). This assumption simplifies the comparative static bounds and offers useful intuition before going to a more general case. allowing for non-local direct effects.

Proposition 1. (Comparative Static under Local Direct Effects). *Suppose assumption 1 holds, and $\forall i, j \neq i : \nabla^x_{ij} = 0$, then $\forall i$*
i*) For the own-effect*

$$\left|\frac{\partial y_i}{\partial x_i}\right| \geq \frac{|\nabla^x_{ii}|}{|\nabla_{ii}|} \cdot \frac{1}{1+\delta_i\delta^*} \tag{8}$$

ii) Moreover, if $\nabla_{ii} < (>)\, 0$, then

$$sgn\left(\frac{\partial y_i}{\partial x_i}\right) = +\,(-)\; sgn\left(\nabla^x_{ii}\right) \tag{9}$$

iii) For the spillover effect, $\forall j \neq i$

$$\delta^* \cdot \left|\frac{\nabla^x_{jj}}{\nabla^x_{ii}}\right| \left|\frac{\partial y_i}{\partial x_i}\right| \geq \left|\frac{\partial y_i}{\partial x_j}\right| \tag{10}$$

Proof. First note that equation (2) and $\forall i, j \neq i : \nabla^x_{ij} = 0$ imply

$$\frac{\partial y_i}{\partial x_j} = \left\{\nabla^{-1}\right\}_{ij} \nabla^x_{jj} \tag{11}$$

Part i): Immediately follows from applying the bound from equation (7) to the absolute of equation (11) under $j = i$. Part ii): $\text{sgn}\left(\frac{\partial y_i}{\partial x_i}\right) = \text{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right)\text{sgn}\left(\nabla^x_{ii}\right) = \text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\nabla^x_{ii}\right)$ where the first equality follows from applying the sign operator to equation (11) under $j = i$, and the second equality follows from

$$\text{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right) = \frac{\text{sgn}\left(\det \nabla_{ii}\right)}{\text{sgn}\left(\det \nabla\right)} = \frac{\text{sgn}\left(\prod_{j\in\mathcal{N}\setminus i}\nabla_{jj}\right)}{\text{sgn}\left(\prod_{j\in\mathcal{N}}\nabla_{jj}\right)} = \text{sgn}\left(\nabla_{ii}\right)$$

the first equality is implied by the cofactor form of the inverse, the second equality applied lemma 1 to both ∇ and its i^{th} principal sub-matrix (which is also diagonally dominant). Part iii) for $j \neq i$: $\left|\frac{\partial y_i}{\partial x_j}\right| = \left|\left\{\nabla^{-1}\right\}_{ij}\right|\left|\nabla^x_{jj}\right| \leq \delta^*\left|\left\{\nabla^{-1}\right\}_{ii}\right|\left|\nabla^x_{jj}\right| = \delta^*\left|\frac{\partial y_i}{\partial x_i}\right|\left|\nabla^x_{ii}\right|^{-1}\left|\nabla^x_{jj}\right|$ where the first equality follows from equation (11). The second (in)equality follows from corollary 1.i). The third equality follows from using equation (11) under $j = i$. Rearranging yields equation (10). □

[9] An alternative interpretation is that $\mathrm{d}\tilde{x}_k \equiv \sum_j \frac{\nabla^x_{kj}}{\nabla^x_{kk}}\mathrm{d}x_j$ is itself the exogenous shock. For example, for comparative statics with respect to market access Bartelme (2015), to average neighbor characteristics (see Jackson and Zenou (2015) pg 123 for a discussion), or shift-share instruments Adão et al. (2019).

where the sign operator $\operatorname{sgn}(a)$ returns the sign of scalar a, i.e. $\operatorname{sgn}(a) = \pm 1$ if $a \gtrless 0$. We discuss each part of the proposition in turn.

Part i). This provides a lower bound on the absolute value of own-effect comparative static: the effect of a shock to a node on the state of the same node, $\frac{\partial y_i}{\partial x_i}$. Importantly note that this bound does not require a matrix inversion of the Jacobian to calculate, thus providing potentially useful information.

Intuition into this bound can be understood as follows. Rewriting the bound, and including an intermediate step

$$\left|\frac{\partial y_i}{\partial x_i}\right| = |\nabla^x_{ii}| \left|\left\{\nabla^{-1}\right\}_{ii}\right| \geq \overbrace{|\nabla^x_{ii}|}^{\textit{direct effect}} \cdot \underbrace{\frac{1}{|\nabla_{ii}|}}_{\textit{zero feedback between nodes}} \cdot \overbrace{\frac{1}{1+\delta_i\delta^*}}^{\textit{attenuation}}$$

The first term is the direct effect of the shock, before the action of any feedback. The second term is equivalent to the feedback in the special case where there is no feedback between nodes. That is, a change in y_i does not cause any endogenous response in outcomes in other nodes, $y_{j\neq i}$, only feedback within the node i on y_i. In this special case, the Jacobian is a diagonal matrix, and therefore its matrix inverse is simply the reciprocal of the diagonal elements in the non-inverted matrix, $\{\nabla^{-1}\}_{ii} = \frac{1}{\nabla_{ii}}$. This describes the feedback operating within the same node only.

Once we bring back the feedback between nodes, the comparative static may be greater or less than this special case depending on whether the feedback configuration — which depends in particular on the sign of the elements ∇_{ij} — causes amplification or attenuation relative to this special case. Because nothing is assumed in proposition 1 about the Jacobian beyond the degree, and therefore nothing about the sign, the bound must be valid in both cases of amplification or attenuation. Thus, with equation (8) being a lower bound in absolute value, it must reflect the most conservative case: the case with maximal attenuation, relative to the special case of no feedback between nodes. The third term, $(1+\delta^*\delta_i)^{-1} \in [0.5, 1]$, turns out to be the factor that corresponds to this case. An interesting feature is that only information on the own and maximal degree, δ_i and δ^*, is required to form this attenuation relative to the no feedback between nodes case. Any further information about which nodes are linked or to what extent, is not required.

To gain more understanding of this bound, we've used simulations to analyze how close the bound is to the exact expression as a function of the maximal degree δ^*. Setting all direct effects to be one, $\nabla^x = I$ (identity matrix), for simplicity, we use $\frac{1}{|\nabla_{ii}|} \cdot \frac{1}{1+\delta^*\delta_i} \Big/ \left|\frac{\partial y_i}{\partial x_i}\right| \in (0, 1]$

as our measure of how tight the bound is (1 corresponds to maximally tight). The results are in figure 1. We've randomly simulated many diagonally dominant Jacobians in $N = 3$ and binned them according to their maximal degree.[10] In each bin, a box plot is presented representing the distribution of tightness within the bin.

We see that as the maximal degree increases, the tightness of the bound in proposition 1 part i) tends to fall. This seems intuitively right. The bound does not use all the information about the Jacobian, and therefore about the feedback in the system. This information loss is greater the greater the feedback is — i.e. the higher δ^* is — therefore the bound is less tight the more feedback there is. This all said, although in the tails the bound tightness decreases substantially (figure 1 suggests almost a one-to-one linear relationship), the median reduction is much more modest, with the bound being about 70% of the exact expression even at $\delta^* = 0.95$.

Finally, note that due to $\delta_i \in [0, 1)$, an even weaker bound on the comparative is implied by proposition 1 part i)

$$\left|\frac{\partial y_i}{\partial x_i}\right| \geq \frac{|\nabla_{ii}^x|}{2\,|\nabla_{ii}|} \tag{12}$$

That is, assuming nothing on the feedback between nodes other than that the Jacobian is diagonally dominant, the local effect comparative static is bounded below by one half of the local effect value in the case of zero feedback between nodes (all in absolute value).

This is a potentially powerful result in applications where complete knowledge of the Jacobian is infeasible (which is common for example in network settings as bilateral network data is costly to collect) or where one doesn't wish to make strong symmetry assumptions in the underlying model.

Part ii). Equation (9) provides the sign of the comparative static under the added assumption that the sign of the corresponding i^{th} diagonal element of the Jacobian is known. When the i^{th} diagonal is negative (positive), the sign of the comparative static $\frac{\partial y_i}{\partial x_i}$ is the same (opposite) sign as the direct effect, ∇_{ii}^x.

Intuition regarding this is as follows. Part ii) tells us that the resulting endogenous change in the state variable y_i moves in the same direction as the shock x_i if the feedback within node i is negative, $\nabla_{ii} < 0$ (and under the assumption the Jacobian is diagonally dominant). To understand why, consider an increase in x_i and lets suppose the direct effect is positive, $\nabla_{ii}^x > 0$. Then, the direct effect of the shock causes f_i to increase. In equilibrium, $f_i = 0$, thus, we require the states $\mathbf{y}$ to adjust endogenously such that f_i ultimately falls back down.

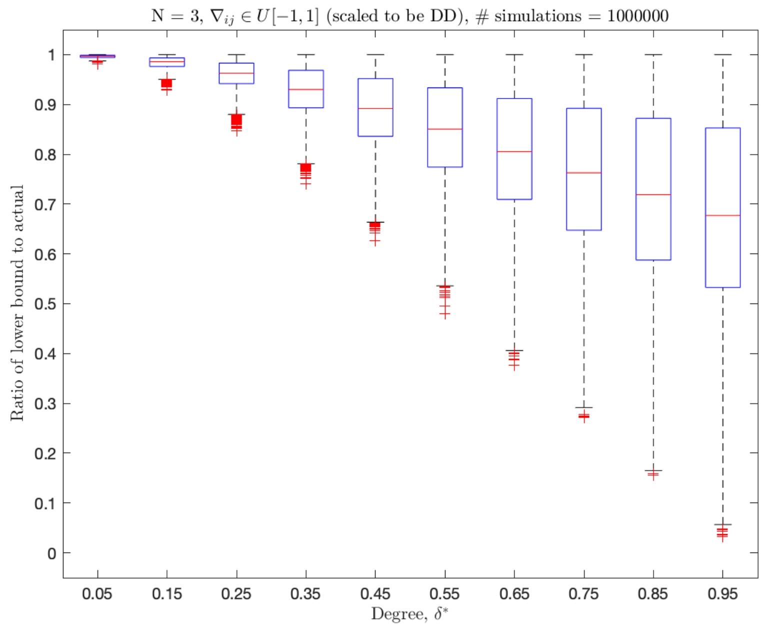
[10] Specifically, $\nabla_{ij} \in U[-1, 1]$ (uniform distribution) and then scaled so $\sum_i \nabla_{ij} \sim U[0, 1]$. We use $N = 3$ to provide maximal coverage of the simulations over the support of ∇.

Ignoring the endogenous responses to $y_{j\neq i}$ for a moment. If $\nabla_{ii} < 0$, then an increase in y_i, and hence movement of y_i in the same direction as x_i, would bring f_i back down, and thus restore equilibrium. This is what equation (9) tells us: $\nabla_{ii} < 0$ implies y_i moves in the same direction as x_i. But what about the effect of endogenous changes to $y_{j\neq i}$? Diagonal dominance effectively imposes that feedback from the node i onto itself has the greatest effect on f_i. In fact, so much so that its effect cannot be overturned by the feedback between nodes. Thus we can effectively ignore the changes in $y_{j\neq i}$ when determining the sign of the change in y_i. The intuition above therefore continues to hold even when we allow for the resulting endogenous changes in $y_{j\neq i}$.

Part iii). Equation (10) implies that the magnitude of the spillover effects — the effect of a shock to a node $j \neq i$ on the state of node i, $\frac{\partial y_i}{\partial x_j}$ — is bounded above by the own-effect on node i, scaled by the ratio of the direct effects, $\left|\frac{\nabla^x_{jj}}{\nabla^x_{ii}}\right|$, and by the maximal degree, δ^*.

Intuition for this is as follows. Diagonal dominance of the Jacobian can be understood as feedback between nodes being weaker than feedback within a node. One might intuitively expect therefore, that if the local direct effects of the shock of the same across nodes, $\nabla^x_{jj} = \nabla^x_{ii}$, the effect on the outcome of node i from a shock to itself (own effect), $\frac{\partial y_i}{\partial x_i}$, to be greater than the effect of a shock to some other node j (spillover effect), $\frac{\partial y_i}{\partial x_j}$. The reason being because the feedback effect for the own effect will be greater than for the spillover effect. This is exactly what equation (10) implies. In fact, it is even stronger: the spillover effect is bounded above by the inverse of the maximal degree, $1/\delta^* \geq 1$, times the own-effect.

Figure 1: Tightness of the bound and Degree



Notes. The figure shows the tightness of the bound against with the degree, across many simulations. We see that the bound tightness decreases as the degree increases.

4.2 Non-Local Direct Effects

Proposition 2 presents results in the more general case of non-local direct effects, $\exists i, j \neq i : \nabla^x_{ij} \neq 0$. In doing so, we still need some restriction on this matrix, which is given by equation (13), as described below.

Proposition 2. (Comparative Static under Non-Local Direct Effects). *Suppose assumption 1 holds. For a given i and j, if*

$$\left|\nabla^x_{ij}\right| \geq \delta^* \sum_{k \neq i} \left|\nabla^x_{kj}\right| \tag{13}$$

then

$$\left|\frac{\partial y_i}{\partial x_j}\right| \geq \frac{\left|\nabla^x_{ij}\right| - \delta^* \sum_{k \neq i} \left|\nabla^x_{kj}\right|}{\left|\nabla_{ii}\right| (1 + \delta_i \delta^*)} \tag{14}$$

Moreover, if $\nabla_{ii} < (>)\, 0$, then

$$sign\left(\frac{\partial y_i}{\partial x_j}\right) = +\,(-)\; sign\left(\nabla^x_{ij}\right) \tag{15}$$

Proof. Equation (14) follows from

$$\left|\frac{\partial y_i}{\partial x_j}\right| = \left|\{\nabla^{-1}\}_{ii}\right| \left|\nabla^x_{ij} + \sum_{k\neq i} \frac{\{\nabla^{-1}\}_{ik}}{\{\nabla^{-1}\}_{ii}} \nabla^x_{kj}\right| \geq \left|\{\nabla^{-1}\}_{ii}\right| \left(\left|\nabla^x_{ij}\right| - \delta^* \sum_{k\neq i} \left|\nabla^x_{kj}\right|\right)$$

where the first equality follows from equation (2), the second (in)equality uses corollary 1.i) and equation (13). Applying corollary 1.ii then yields equation (14). Equation (15) follows from

$$\text{sign}\left(\frac{\partial y_i}{\partial x_j}\right) = \text{sign}\left(\{\nabla^{-1}\}_{ii}\right) \text{sign}\left(\nabla^x_{ij}\right) \text{sign}\left(\left|\nabla^x_{ij}\right| + \text{sign}\left(\nabla^x_{ij}\right) \sum_{k\neq i} \frac{\{\nabla^{-1}\}_{ik}}{\{\nabla^{-1}\}_{ii}} \nabla^x_{kj}\right)$$

Corollary 1.i) and equation (13) imply the final term is always positive. The remainder of the proof then follows the proof of proposition 1.ii).

□

The proposition can be used to derive a bound between any two nodes i and j for which the direct effects matrix, ∇^x, satisfies equation (13). This restriction essentially ensures that the direct effect of the shock from j onto i, ∇^x_{ij}, is most "important" in determining the overall effect of the shock from j onto i, i.e. the comparative static $\frac{\partial y_i}{\partial x_j}$. That is, the effect on i stemming from direct effects on $k \neq i$, ∇^x_{kj}, are sufficiently small. The less feedback there is in the system — a smaller δ^* — the less binding this condition is. This is because the only way an effect from j onto $k \neq i$ can affect the outcome in i is due to endogenous feedback between nodes. Correspondingly, in the limit of no feedback between nodes, $\delta^* = 0$, the restriction on ∇^x — equation (13) — becomes vacuous. Note also that this restriction is trivially satisfied for $j = i$ when there are no direct effects between nodes, $\nabla^x_{i,j\neq i} = 0$ — the case in proposition 2.

Under this restriction, the bound given by proposition 2 in equation (14) can essentially be understood as the same bound in proposition 1 equation (8), but with an "effective" direct effect of $\left|\nabla^x_{ij}\right| - \delta^* \sum_{k\neq i} \left|\nabla^x_{kj}\right|$. This is the direct effect from j onto i, $\left|\nabla^x_{ij}\right|$, net of the possible effects on the outcome in i stemming from the direct effects onto $k \neq i$, $\delta^* \sum_{k\neq i} \left|\nabla^x_{kj}\right|$. The δ^* appears due to this term reflecting the endogenous feedback as discussed in the paragraph above. Under some parameterizations, these direct effects via $k \neq i$ may reinforce the effect from j on i. However, because the proposition makes no assumption on the signs of the interactions, the bound reflects the most conservative case and therefore these terms enter with a minus sign.

Equation (15) is the same as equation (9) in proposition 1. That is to say, even once we allow for non-local effects, assuming they satisfy equation (13), a sufficient condition for the sign of the comparative static is the same as in the non-local direct effects case. The intuition is similar to that described in section 4.1. Under the restriction that the corresponding direct effect is dominant — equation (13) — and that feedback is bounded because of diagonal dominance, the direct effects on other nodes $k \neq i$, and the resulting feedback from them, are not large enough to overturn the sign of the direct effect on i.

4.3 Summary

In both propositions 1 and 2, the message we deliver is the same. The comparative static can be partially identified (i.e. a bound) without needing to invert the Jacobian.

This has two implications. The first is that the analytic form of the bound is simpler than the exact expression, therefore providing a tool to derive qualitative properties of the comparative static. The second is that all the information contained in the Jacobian is not used in the bound — the starkest demonstration being equation (12). This therefore provides a method of identification under incomplete information of the Jacobian.

In the next section, we provide applications demonstrating these implications.

5 Applications

5.1 Network Games

We first demonstrate our results in a standard network game with linear best replies (see e.g. Ballester et al. (2006), Bramoullé et al. (2014)). We allow for either strategic complements or substitutes.

Player i choses effort $y_i \in \mathbb{R}$ to maximize a linear-quadratic utility function

$$u_i = a_i y_i - \frac{1}{2} y_i^2 + \phi \sum_{j=1}^{n} G_{ij} y_i y_j$$

$a_i > 0$ is the private benefit received by exerting effort; this will serve as the exogenous shock we'll take the comparative static with respect to. The quadratic term in effort of player i and every other player j implies there is strategic interaction between players when maximizing utility. $G_{ij} \in \{0, 1\}$ is the adjacency matrix, with value one if agents i and j are connected. We assume $G_{ii} = 0$ as standard (there is no interaction with oneself). $\phi \in \mathbb{R}$ scales the magnitude of strategic interaction, with $\phi > 0$ indicating strategic complements, and $\phi < 0$ strategic substitutes.

The first order condition of $\max_{y_i} u_i$ yields player i's best reply

$$y_i = a_i + \phi \sum_{j=1}^{n} G_{ij} y_j \tag{16}$$

The Nash equilibrium is the $\mathbf{y}$ such that all players' best replies are satisfied. The best replies correspond to the equations of state, equation (1)

$$f_i(\mathbf{y}, \mathbf{a}) \equiv y_i - a_i - \phi \sum_{j=1}^{n} G_{ij} y_j = 0$$

The Jacobian in this model is

$$\nabla_{ij} \equiv \frac{\partial f_i}{\partial y_j} = I_{ij} - \phi G_{ij}$$

The Jacobian satisfies diagonal dominance (assumption 1) if

$$1 > |\phi| \sum_j G_{ij} \tag{17}$$

where we used $\sum_{j \in \mathcal{N} \setminus i} G_{ij} = \sum_j G_{ij}$. By the Gershgorin Circle Theorem, equation (17) implies $1 > |\phi| \, |\lambda_{max}(G)|$, which is precisely the condition for equilibrium uniqueness in Ballester et al. (2006) theorem 1 under strategic complements, and implied by the condition in Bramoullé et al. (2014) section II under strategic substitutes.[11]

The degree of player i is

$$\delta_i = |\phi| \sum_j G_{ij} \tag{18}$$

i.e. proportional to the number of players that i is connected to. This is precisely the concept of "degree centrality" in the network science literature (see e.g. Bramoullé et al. (2016) chapter 11).

The diagonal of the Jacobian is always positive $\forall i : \nabla_{ii} = 1 > 0$. The direct effects matrix is

$$\nabla^x_{ij} \equiv \frac{\partial f_i}{\partial a_j} = -I_{ij}$$

Thus, we are in the realm of local direct effects — proposition 1. The comparative statics for the effect of an increase in private benefit a_i on the effort of the same agent y_i, is given by

$$\frac{\partial y_i}{\partial a_i} = -\left\{ (I - \phi G)^{-1} \right\}_{ii} \tag{19}$$

[11] Bramoullé et al. (2014) present the sufficient condition $1 > |\phi| \, |\lambda_{min}(G)|$. Note that this is implied by $1 > |\phi| \, |\lambda_{max}(G)|$.

The bound implied by proposition 1 is

$$\frac{\partial y_i}{\partial a_i} \geq \frac{1}{1 + |\phi|^2 \left(\sum_j G_{ij}\right) \cdot \left(\sum_j G_{i^*j}\right)} \tag{20}$$

$$\geq \frac{1}{2} \tag{21}$$

where $i^* = \arg\max_i \delta_i$. Equation (20) is the signed bound implied by parts i) and ii) of proposition 1. We see it only depends on ϕ and the number of network nodes — network degree — i is connected to, and the maximal degree in the network.

These relationships permit partial identification of the comparative static (a bound) with only partial information on the network: beyond ϕ and the degree centrality, information on who is connected to whom is irrelevant. Moreover, they have considerably more tractable functional form than the exact relationship in equation (19). This allows one to derive qualitative properties of the comparative statics very simply. For example, if one wanted to deduce a value of ϕ that guarantees $\frac{\partial y_i}{\partial a_i} \geq x$ for some $x > 0$, equation (20) provides an answer to this: $|\phi| \geq \frac{1-x}{x\delta_i\delta^*}$.

Equation (21) provides an even simpler, though weaker, bound from using equation (17). This tells us that for any adjacency matrix subject to equation (17) (which, recall, implies diagonal dominance of the Jacobian), the local effect comparative static, $\frac{\partial y_i}{\partial a_i}$, is always bounded below be $\frac{1}{2}$. The intuition for this bound is as follows. With no feedback between nodes $\forall i, j : G_{ij} = 0$, the comparative static is $\frac{\partial y_i}{\partial a_i} = 1$. With network feedback, this bound can be attenuated if there is negative feedback — referred to as strategic substitutes in network model. Diagonal dominance places a bound on the amount of feedback generally, and therefore also on the amount of attenuation possible. It implies that negative feedback can only at most reduce the comparative static by half relative to the case of no feedback between nodes.

5.2 Oligopoly

We next demonstrate our results in a canonical oligopoly model with differentiated products Dixit (1986). We assume Cournot competition, and otherwise mostly follow the assumptions version in Galeotti et al. (2022), as they permit rich heterogeneity in interdependence of demand, and simplifying assumptions conducive for a clear exposition.[12]

There are N firms each producing one good. Firms engage in a Cournot competition,

[12] Galeotti et al. (2022) assumes Bertrand competition. We use Cournot as the inverse demand schedules estimated in Pellegrino (2019) satisfy diagonal dominance.

simultaneously choosing quantities $\mathbf{q} = \{q_i\}_{i\in\mathcal{N}}$. The profit function firm i faces is

$$\pi_i(\mathbf{q}, c_i) = (p_i - c_i)\, q_i \tag{22}$$

The firm faces linear cost where c_i is the (constant) marginal cost, and assumed exogenous. We will consider comparative statics with respect to this. The firm chooses quantity q_i to maximizes profit subject to demand

$$\max_{q_i} \pi_i(\mathbf{q}, c_i)\,, \text{ s.t. } p_i = p_i^D(\mathbf{q}) \tag{23}$$

where $p_i^D(\mathbf{q})$ is the (inverse) demand function faced by firm i. Following Galeotti et al. (2022), we assume that demand is linear (sufficiently close to the equilibrium) and that the own-price effect of demand is negative[13]

$$\forall i: \quad \frac{\partial p_i^D(\mathbf{q})}{\partial q_i} < 0 \tag{24}$$

Firm i's optimal quantities are characterized by the first order conditions of equation (23)

$$\frac{\partial \pi_i(\mathbf{q})}{\partial q_i} = \underbrace{(p_i - c_i) + \frac{\partial p_i^D}{\partial q_i} q_i}_{\equiv f_i(\mathbf{p},\mathbf{c})} = 0 \tag{25}$$

Note that equation (24) implies the second order conditions ($\frac{\partial^2 \pi_i}{\partial^2 q_i} = 2\frac{\partial p_i^D}{\partial q_i} < 0$) are satisfied.

Equation (25) corresponds to the equations of state in our framework, equation (1), with $\mathbf{q}$ the endogenous state variable, and $\mathbf{c}$ the exogenous shock.

The Jacobian, accordingly, is

$$\nabla_{ij} \equiv \frac{\partial f_i(\mathbf{p},\mathbf{c})}{\partial q_j} = \frac{\partial p_i^D}{\partial q_j}(1 + I_{ij})$$

The Jacobian satisfies diagonal dominance (assumption 1) if

$$\left|\frac{\partial p_i^D}{\partial q_i}\right| > \frac{1}{2}\sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial p_j^D}{\partial q_i}\right|$$

That is, the own-price effect of (inverse) demand is sufficiently greater than the cross-price

[13] One could further assume demand to be consistent with a utility maximizing representative consumer as in Galeotti et al. (2022), though we do not require this.

effects of (inverse) demand. The direct effect matrix is

$$\nabla^x_{ij} \equiv \frac{\partial f_i(\mathbf{p}, \mathbf{c})}{\partial c_j} = -I_{ij}$$

and thus we are in the realm of proposition 1. Now to the comparative statics. Consider the effect of a shock to firm i's marginal cost on its own quantity produced. Using equation (2), this is given by

$$\frac{\partial q_i}{\partial c_i} = \left\{ \left[\frac{\partial p^D}{\partial q} \odot (\mathbf{11}' + I) \right]^{-1} \right\}_{ii} \tag{26}$$

where $\odot$ is the Hadamard (element-wise) product, and $\mathbf{1}$is the vector of ones. Although the comparative static only concerns the marginal cost and quantity of firm i, the comparative static depends on the own- and cross-price effects of the (inverse) demand for all firms, $\forall i, j : \frac{\partial p_j^D}{\partial q_i}$, due to the matrix inverse.

The reason for this, intuitively, is as follows. The change in firm i's cost directly causes firm i to update its quantity, q_i. This change in quantity causes the residual demand facing all other firms to change, and thus they each update their quantity produced. This update in quantity likewise causes firm i to update its quantity, now for a second time. This happens ad infinitum until a fixed point in quantity is arrived.[14] The response in residual demand to changes in quantity of other firms is precisely described by the price effects $\frac{\partial q_j^D}{\partial p_i}$, and thus the resulting change in the quantity of firm i after all rounds of updating depends on the whole matrix of price effects.

The consequence is a demanding identification problem: we need to identify all N^2 bilateral price effects even if we are only interested in the effect of a cost shock on the quantity of the same firm, $\frac{\partial q_i}{\partial c_i}$.[15] Moreover, the dependence of this comparative static of this on, say, the firm's own-price effect, $\frac{\partial p_i^D}{\partial q_i}$, is hard to discern due to the highly non-linear dependence through the matrix inverse.

The results of this paper provide a tool to make progress on both these challenges. Applying parts i) and ii) of proposition 1, and using that $\forall i : \delta_i \leq 1$, we find that

$$\frac{\partial q_i}{\partial c_i} \leq -\frac{1}{2\left|\frac{\partial p_i^D}{\partial q_i}\right|}$$

i.e. the change in output is partially identified with knowledge only of the own-(inverse) price effect of demand, $\frac{\partial p_i^D}{\partial q_i}$. Moreover, we know that for flatter own-price effects, $\left|\frac{\partial p_i^D}{\partial q_i}\right| \downarrow$,

[14] Of course, there is no dynamics in the model, and instead this all happens simultaneously.

[15] The alternative method is some form of dimensionality reduction, such as parameterizing the elasticities as function of $K \ll N$ observable characteristics (Berry (1994); Berry et al. (1995); Pellegrino (2019)).

the fall in own output q_i is expected to be greater. This dependence is hard to discern from equation (26), yet implied immediately from the tools in the paper.

Finally, we can use the tools in this paper to bound other endogenous variables in the model, for example the effect on profits of firm i.

$$\begin{aligned}
\frac{\partial \pi_i}{\partial c_i} &= \sum_j \frac{\partial \pi_i}{\partial q_j}\frac{\partial q_j}{\partial c_i} + \frac{\partial \pi_i}{\partial c_i} \\
&= q_i \sum_{j\neq i} \frac{\partial p_i^D}{\partial q_j}\left\{\nabla^{-1}\right\}_{ji} - q_i \\
&= -q_i \nabla_{ii}\left\{\nabla^{-1}\right\}_{ii} \\
&\leq -\frac{q_i}{2}
\end{aligned} \tag{27}$$

The first line is a total differentiation of profits, equation (22). The second line used $\frac{\partial \pi_i}{\partial q_i} = 0$ by the FOC, and inserted in the values of each remaining derivative. The third line used $\sum_{j\neq i} \frac{\partial p_i^D}{\partial q_j}\left\{\nabla^{-1}\right\}_{ji} = \sum_j \nabla_{ij}\left\{\nabla^{-1}\right\}_{ji} - \nabla_{ii}\left\{\nabla^{-1}\right\}_{ii} = 1 - \nabla_{ii}\left\{\nabla^{-1}\right\}_{ii}$. The fourth line applies the result in corollary 1.ii) with $\delta_i \leq 1$ directly to bound, $|\left\{\nabla^{-1}\right\}_{ii}| \geq \frac{1}{|\nabla_{ii}|}\frac{1}{1+\delta_i\delta^*} \geq \frac{1}{2|\nabla_{ii}|}$.

Equation (27) is a very succinct result. We find that from an increase in marginal cost c_i the profits of the same firm will fall by at least $q_i/2$ in absolute value. This can be understood as follows. The total cost of the firm is $c_i q_i$, and thus increases proportionally by q_i from an increase in c_i. The change in profits may be more or less than q_i depending on what happens to revenue $p_i q_i$. Revenue will fall if the cost increase causes consumers to on net substitute away from i to other firms' goods — and the opposite if consumers substitute towards i. Both substitution patterns are permissible as we've not assumed anything about the cross-price effects $\frac{\partial p^D}{\partial q}$ other than diagonal dominance — in particular we've assumed nothing about whether these are complements or substitutes. However, what diagonal dominance does imply is that the (negative) own-price effects are sufficient great that any increase in revenue due to cross-price effects between firms cannot raise profits more than half of the direct effect on cost, ie $q_i/2$. i.e. these cross-price effects are sufficiently weak relative to the own-price effect that this bound is always guaranteed.

6 Conclusion

In this paper we revisit an old inquiry in economics: what can we deduce about comparative statics while making as few assumptions as possible? This is often referred to as qualitative or nonparametric economics Bassett et al. (1967), Hale et al. (1999). We offer new results moving the frontier in this subject. In particular, we use the established condition of diagonal dominance of the Jacobian, and derive new implications from this. We show that this implies useful properties on the inverse of the Jacobian that can be used to provide bounds on the comparative statics.

The value of this result is that the bounds are simpler than that the exact relationship. This bounds are identified using only a subset of the information in the Jacobian, therefore permitting partial identification of the comparative static when full knowledge of the Jacobian is too costly or infeasible. Moreover, due to the tractability of the bounds, they potentially permit easier characterization of the qualitative properties of the comparative static, and therefore new insight into the underlying economic mechanisms.

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A Appendix

A.1 Proof Theorem 1

We will prove a slighter stronger result than written in theorem 1.

Theorem. (Minors of Weakly Diagonally Dominant Matrices) *Suppose that the $N \times N$ real matrix, ∇, is weakly column diagonal dominant,*

$$\forall i \in \{1, ..., N\}: \quad |\nabla_{ii}| \geq \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}| \tag{28}$$

then,

$$\forall j, i \neq j: \quad |\det \nabla_{-j,-i}| \leq \delta^* |\det \nabla_{-i,-i}| \tag{29}$$

with the inequality being strict in the case of strict diagonal dominance (the case presented in theorem 1).

Proof. Without loss of generality, suppose

$$\forall i: \nabla_{ii} \geq 0 \tag{30}$$

This is because all the minors of

$$\text{diag}\left[\text{sign}\left(\nabla_{11}\right), \cdots, \text{sign}\left(\nabla_{NN}\right)\right] \nabla \tag{31}$$

in absolute value are equal to the ones of ∇ and the matrix in equation (31) has a nonnegative diagonal. Note that equation (30), combined with weak diagonal dominance, implies $\forall i$: $\det \nabla_{-i,-i} \geq 0$ (see the proof of proposition 4.1.ii).

It suffices to consider $i = 1$, i.e. to prove that

$$\forall j \neq 1: \quad |\det \nabla_{-j,-1}| \leq \delta^* |\det \nabla_{-1,-1}| \tag{32}$$

Let C_r be the rth row of ∇ without the first entry

$$C_r = (\nabla_{r2}, \cdots, \nabla_{rN}) \tag{33}$$

Case 1: $\det \nabla_{-j,-1} \geq 0$: using the notation from equation (33), we have

$$\nabla_{-j,-1} = \left(C_1', \cdots C_{j-1}', C_{j+1}', \cdots, C_N'\right)$$

and

$$\nabla_{-1,-1} = \left(C_2', \cdots, C_N'\right)$$

where the prime indicates transpose. Therefore

$$\left|\det \nabla_{-j,-1}\right| \leq \delta^* \left|\det \nabla_{-1,-1}\right|$$

is equivalent to

$$\begin{aligned} 0 &\leq \delta^* \det\left(C_2', \cdots, C_N'\right) - \det\left(C_1', \cdots C_{j-1}', C_{j+1}', \cdots, C_N'\right) \\ &= \det \underbrace{\left(C_2', \cdots C_{j-1}', \delta^* C_j + (-1)^j C_1, C_{j+1}', \cdots, C_N'\right)}_{\equiv M} \end{aligned} \tag{34}$$

Now, $\det M \geq 0$ is true, and thus equation (32) proved under this case, because M has i) nonnegative diagonals and ii) is diagonally dominant (the combination implying M has nonnegative determinant). To see both of these conditions are satisfied, let's write out the matrix

$$M = \begin{pmatrix} \nabla_{2,2} & \cdots & \nabla_{j-1,2} & \delta^* \nabla_{j,2} + (-1)^j \nabla_{1,2} & \nabla_{j+1,2} & \cdots & \nabla_{N,2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \delta^* \nabla_{jj} + (-1)^j \nabla_{1,j} & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \nabla_{2,N} & \cdots & & \delta^* \nabla_{j,N} + (-1)^j \nabla_{1,N} & & \cdots & \nabla_{N,N} \end{pmatrix}$$

We see that M has i) nonnegative diagonal because for rows $\forall r \in \mathcal{N} : \nabla_{rr} \geq 0$ by equation (30) and

$$\begin{aligned} \delta^* \nabla_{jj} + (-1)^j \nabla_{1,j} &\geq \delta_j \nabla_{jj} + (-1)^j \nabla_{1,j} \\ &\geq \sum_{i \in \mathcal{N} \setminus j} \left|\nabla_{ij}\right| + (-1)^j \nabla_{1,j} \\ &\geq \sum_{i \in \mathcal{N} \setminus \{1,j\}} \left|\nabla_{ij}\right| \\ &\geq 0 \end{aligned}$$

using the definition of δ^* and δ_i — equations (4) and (3). We see that ii) M is (row) diagonally dominant because (sufficient to show for the first row)

$$\underbrace{\nabla_{2,2}}_{\text{diagonal element}} - \underbrace{\left(\sum_{i \in \mathcal{N} \setminus \{1,2,j\}} |\nabla_{i,2}| + \delta^* \nabla_{j,2} + (-1)^j \nabla_{1,2} \right)}_{\text{sum of off-diagonal elements}} \geq \nabla_{2,2} - \sum_{i \in \mathcal{N} \setminus j} |\nabla_{i,2}|$$
$$\geq 0$$

by diagonal dominance of ∇ — equation (28).

<u>**Case 2**: $\det \nabla_{-j,-1} < 0$</u>: Note that

$$\begin{aligned} |\det \nabla_{-j,-1}| &= -\det \nabla_{-j,-1} \\ &= \det \left(-C_1', \cdots C_{j-1}', C_{j+1}', \cdots, C_N' \right) \end{aligned}$$

The argument will be the same as before except applied to the matrix

$$\tilde{M} = \left(C_2', \cdots C_{j-1}', \delta^* C_j + (-1)^{j+1} C_1, C_{j+1}', \cdots, C_N' \right)$$

Thus, equation (32) is implied under both cases and therefore generally. Therefore, the theorem is proved. □

A.2 Determinant Sign of Diagonally Dominant Matrices

Lemma 1. (Determinant Sign of Diagonally Dominant Matrices) *Suppose that the $N \times N$ real matrix, ∇, is column diagonal dominant,*

$$\forall i \in \{1, ..., N\}: \quad |\nabla_{ii}| > \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}| \tag{35}$$

then,

$$sgn\left(\det \nabla\right) = sgn\left(\prod_{i=1}^{N} \nabla_{ii}\right) \tag{36}$$

Proof. Define the matrix $\tilde{\nabla}$ implicitly from

$$\nabla = \underbrace{\operatorname{diag}\left[\operatorname{sgn}\left(\nabla_{11}\right), \cdots, \operatorname{sgn}\left(\nabla_{NN}\right)\right]}_{\equiv \Lambda} \tilde{\nabla} \tag{37}$$

note that $\tilde{\nabla}$ inherits diagonal dominance from ∇ because $\left|\tilde{\nabla}_{ij}\right| = |\nabla_{ij}|$, and $\tilde{\nabla}$ has non-negative diagonals $\forall i : \tilde{\nabla}_{ii} \geq 0$. Thus, by Horn and Johnson (2012) T.6.1.10, we know all eigenvalues of $\tilde{\nabla}$ have positive real part. Therefore $\det \tilde{\nabla} > 0$, and as we'll use below, $\operatorname{sgn}\left\{\det\left\{\tilde{\nabla}\right\}\right\} = 1$.

The matrix Λ defined in equation (37) is diagonal, therefore the determinant of Λ is equal to the product of its diagonal entries.

Thus, taking the determinant of ∇ via equation (37),

$$\begin{aligned}\det \nabla &= \det\left\{\operatorname{diag}\left[\operatorname{sign}\left(\nabla_{11}\right), \cdots, \operatorname{sign}\left(\nabla_{NN}\right)\right]\right\} \cdot \det\left\{\tilde{\nabla}\right\} \\ &= \prod_{i=1}^{N} \operatorname{sgn}\left(\nabla_{ii}\right) \cdot \det\left\{\tilde{\nabla}\right\}\end{aligned}$$

Now applying the sign operator

$$\operatorname{sgn}\left(\det \nabla\right) = \operatorname{sgn}\left\{\prod_{i=1}^{N} \operatorname{sgn}\left(\nabla_{ii}\right)\right\} \cdot \underbrace{\operatorname{sgn}\left\{\det\left\{\tilde{\nabla}\right\}\right\}}_{=1}$$

Thus, we have equation (37). $\square$