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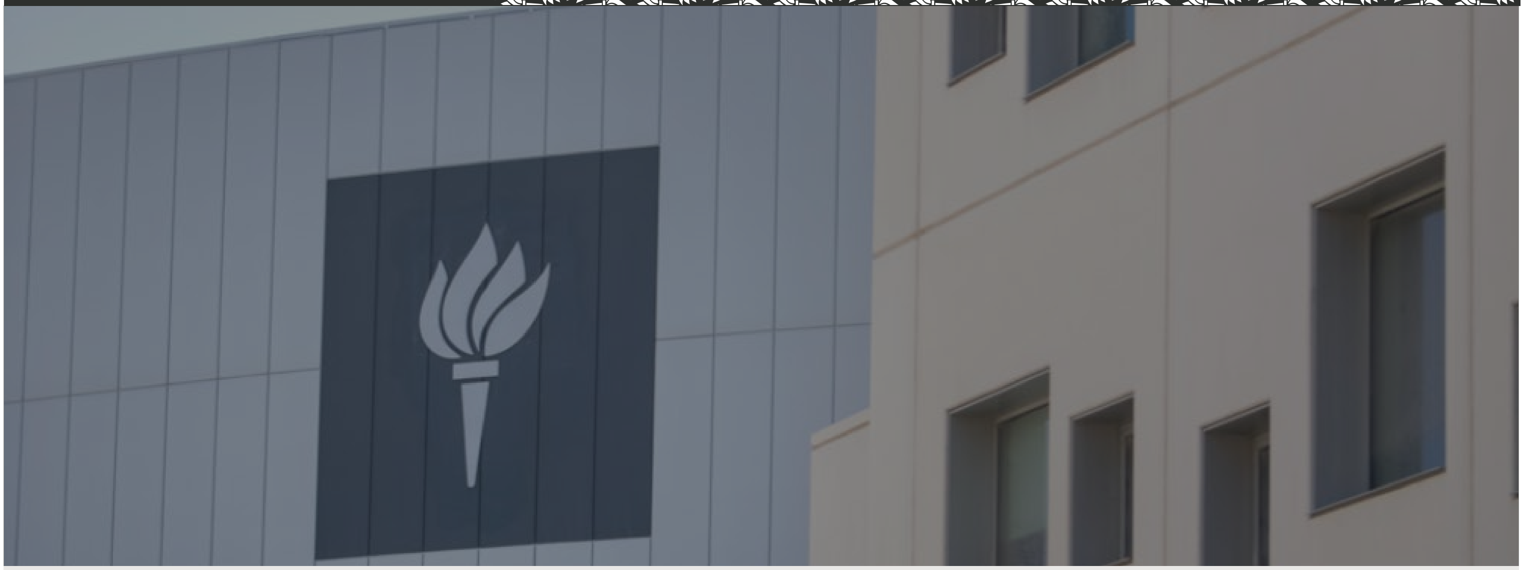

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Allocating Positions Fairly: An Auction and its Relationship to the Shapley Value*

Matt Van Essen[†] John Wooders[‡]

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Abstract

We study the problem of fairly allocating heterogenous items, priorities, positions, or property rights to participants with equal claims, in an incomplete information environment. We introduce a dynamic auction for solving this problem and we characterize both Bayes Nash equilibrium and "maxmin perfect" play. Building on these characterizations we show that: (i) equilibrium play converges to maxmin perfect play as bidders become infinitely risk averse, and (ii) each bidder obtains his Shapley value when every bidder follows his maxmin perfect strategy. Hence, the equilibrium allocation converges to the Shapley value allocation as bidders become more risk averse. Together these results provide both noncooperative and decision theoretic foundations for the Shapley value in an environment with incomplete information.

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1 Introduction

This paper studies the problem of allocating heterogeneous items, priorities, positions, or rights to participants who have equal claims. Examples of this type of problem include allocating items to heirs in an estate, allocating the priority of service in a queue, allocating the position of an ad on a webpage or assigning faculty to offices, or allocating fishing rights to different geographical areas. In our environment, participants have unit demands and a common ranking of the items/priorities/positions/rights, which we hereafter simply refer to as "positions." All participants agree that one position is the most desirable, a second position is the next most desirable, and so on. Despite the common ranking of positions, participants vary in the intensity of their preferences and these intensities are private information. The problem is to find an allocation that is efficient, budget balanced, and fair, with those participants receiving more desirable positions compensating the ones receiving less desirable positions.

We introduce a dynamic auction for solving this problem and we characterize its equilibrium when participants (hereafter "bidders") are risk neutral and when they are risk averse. The auction takes place over rounds, where at each round the worst remaining position is allocated via an ascending clock auction. A bidder who drops out is allocated the position at auction in the current round and receives compensation equal to the price at which he dropped, with the compensation paid shared equally among the remaining bidders, all of whom will ultimately obtain better positions. The auction ends when one bidder remains. He receives the most desirable position, but pays compensation to every other bidder. Thus a bidder pays compensation to bidders allocated positions worse than his own and receives compensation from bidders allocated positions better than his own.

We provide general necessary conditions for a bidding strategy to form a symmetric equilibrium in increasing and differentiable strategies. We give closed-form solutions for the unique such equilibrium when bidders are risk neutral and when they are CARA risk averse. We show that bidders drop out earlier, accepting less compensation, as they become more risk averse.

An alternative approach to modelling behavior in our auction is from a decision-theoretic perspective, with each bidder acting to maximize his minimum payoff. We will say a strategy is "maxmin perfect" if it maximizes a bidder's minimum payoff at every history of play. Maxmin perfection is a natural refinement of maxmin for dynamic games. We characterize the unique maxmin perfect strategy. The strategy has a natural fairness interpretation as it calls for a bidder to demand equal shares of the incremental benefits obtained by bidders who are allocated better positions. Thus the demands resemble the Talmudic solution to the well-known contested garment problem.

We have three main results. First, the equilibrium bidding strategy of CARA risk averse bidders converges to the maxmin perfect bidding strategy as bidders become infinitely risk averse. Second, when every bidder follows his maxmin perfect strategy, then each bidder obtains his Shapley value allocation. Our third result, an immediate consequence the first two, is that the equilibrium allocation of the auction coincides with the Shapley value allocation as bidders become infinitely risk averse. Hence our results provide non-cooperative and decision theoretic foundations for the Shapley value in an environment with incomplete information.

Shapley (1953) introduced the notion of a *value* for a cooperative game, now called the Shapley value. The Shapley value is a fundamental solution concept in cooperative game theory with the Shapley allocation often taken as the benchmark for a fair allocation (see Myerson (1977), Roth (1988), Moulin (1992), and Moulin (2004, Chapter 5)). To our knowledge, this paper is the first to provide non-cooperative foundations for the Shapley value in a setting with incomplete information.

Related Literature

Our paper connects to a variety of literatures in non-cooperative and cooperative game theory.

The Assignment Problem: The problem of allocating positions is the assignment problem for the special case where all the players rank assignments in the same way, as is natural for example when assignments correspond to priorities, e.g., first priority, second priority, etc. Both cooperative and non-cooperative solutions to the general assignment problem have been studied. Moulin (1992) shows that the Shapley value has several desirable properties in cooperative models of assignment games.[1]

Early examples of non-cooperative approaches to the assignment problem include Leonard (1983) and Demange, Gale, Sotomayor (1986). Leonard (1983) provides a mechanism for which it is a dominant strategy for each player to report his preferences over assignments truthfully and which implements the efficient assignment; he shows it generates Vickrey-Clark-Groves prices. Demange, Gale, Sotomayor (1986) provide a dynamic auction which implements the efficient assignment. In the context of internet advertising, important papers by Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007) study the use of the generalized second-price sealed-bid auction to allocate positions under complete information. Edelman, Ostrovsky, and Schwarz (2007) study, in addition, a generalized English auction with incomplete information and show that payoffs (both to bidders and to the seller) are the same as in the Vickrey-Clarke-Groves mechanism.

In all these papers, the seller collects the auction revenue. We study, in contrast, a setting where there is no seller and the only payments are transfers between the bidders. Budget balancedness is a fundamental requirement since the positions are the common property of the bidders.

Non-cooperative Foundations of the Shapley Value: In bargaining games with complete information, non-cooperative foundations of the Shapley value have been provided by Gul (1989) and Hart and Mas Colell (1996). Gul (1989) provides a game with bilateral bargaining and the random selection of the proposer and shows that, in the efficient equilibrium of the game, players

[1] The Shapley value is not the only notion of fairness for assignment games. Alkan, Demange, and Gale (1991), for example, study existence of efficient and envy-free allocations in the assignment problem.

receive their Shapley value payoffs in the limit as they become perfectly patient. Hart and Mas Colell (1996) studies a multilateral bargaining game and shows that players receive their Shapley value payoffs in the limit as each player's probability of exogenously exiting from bargaining vanishes. By contrast, we obtain Shapley value payoffs as bidders become infinitely risk averse in an environment with incomplete information.

Bidding Rings, Bankruptcy, and Cost Sharing: The Shapley value also appears in the literature on collusion in auctions. Graham, Marshall, and Richard (1990) shows that bidders receive their Shapley value payoffs in a nested knockout auction when bidding rings are perfectly nested. In their setting, the bidders' values for the item are commonly known and bidders are assumed to remain active in the knockout auction until the bid reaches their value (although this is not equilibrium behavior).[2]

Aumann and Maschler (1985) shows that the solutions provided in the Talmud of three different bankruptcy problems coincide with the nucleoli of the corresponding cooperative games. These solutions are generalizations of the solution to the contested garment problem: "Two hold a garment; one claims it all, the other claims half. Then the one is awarded three-fourths, the other one-fourth." In this solution, the lesser claimant concedes the uncontested half the garment to the greater one, and the remainder is split equally. In our auction, at each round all but the worst remaining position are contested. The maxmin bid at each round can be interpreted as a demand for equal shares of the incremental benefits of the contested positions, and in this respect resembles the solution to the contested garment problem.

Finally, the rules of our auction are reminiscent of serial cost sharing. Moulin and Shenker (1992) studies the problem of allocating costs when agents face a production technology with decreasing returns to scale. It proposes a cost sharing rule in which participants pay equal shares of incremental costs (defined in a particular way) and show that, given this rule,

[2] Littlechild and Owen (1973) obtain the same payoffs when allocating costs to the users of an airport runway.

the game in which the participants announce quantities is dominance solvable and equilibrium has several nice properties.[3] The cost sharing rule is a primitive, part of the description of the game, whereas here the surplus shares are endogenously determined. In our setting, equilibrium demands for compensation can be interpreted as (inflated) demands for equal shares of the incremental benefits of contested positions.

The rest of the paper proceeds as follows: We provide in Section 2 a description of the position allocation problem and we identify the Shapley value of the associated cooperative game. Section 3 describes the compensated position auction and the private values environment. Our equilibrium results are in Section 4, while Section 5 identifies maxmin perfect strategies. Section 6 relates equilibrium, maxmin, and the Shapley value. We conclude with a discussion in Section 7. All proofs are in the Appendix.

2 Allocating Positions Cooperatively – The Shapley Value

$N \geq 2$ positions are to be allocated to N players, one to each, who have equal claims.[4] The positions have inherent values, denoted by $\alpha_1, ..., \alpha_N$, which are commonly known. We order the positions so that $\alpha_1 \geq ... \geq \alpha_N$. Positions may be desirable or undesirable, i.e., we allow a mixture of positive and negative α's.[5] Let $x_1, ..., x_N$ be the profile of player values. In this section it is convenient to order the players so that $x_1 \geq ... \geq x_N$. The payoff

[3] The serial cost sharing rule is similar in spirit to one of the Talmudic procedures described in Aumann and Maschler (1985, p. 203). Specifically, the procedure suggested by Rabad for compensating a seller at an auction where all of the bidders renege on their bids.

[4] This is without loss of generality since, if there are more players than positions, one can create dummy positions, with α's equal to zero, until the number of positions equals the number of players.

[5] Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2017) study a fair division problem when the goods to be divided are a mixture of both goods and bads.

to a player whose value is x and who receives position t is $\alpha_t x$ plus any net transfer he receives. The problem is to efficiently and fairly allocate positions to players, while respecting budget balance.

Cooperative game theory suggests a solution: allocate positions to maximize surplus and make transfers among the players so that each player receives his Shapley value. The Shapley solution is appealing since it the only solution satisfying (i) efficiency, (ii) additivity, (iii) symmetry, and (iv) no surplus to dummy players. For a general characteristic function v, the Shapley value ϕ_i of player i is

$$\phi_i = \sum_{S \subseteq \{1,\ldots,N\}} \frac{(|S|-1)!(N-|S|)!}{N!} \left[v(S) - v(S \backslash \{i\}) \right],$$

where $v(S)$ gives the value of coalition S. Player i's Shapley value can be interpreted as his expected marginal contribution when the grand coalition is formed by adding players, one at a time, in a random order.

We now describe the Shapley solution to the position allocation problem. For any coalition $S \in 2^N$, let $y_1^{(S)}, \ldots, y_{|S|}^{(S)}$ be a rearrangement of the values $\{x_i | i \in S\}$ of the members of S such that $y_1^{(S)} \geq \ldots \geq y_{|S|}^{(S)}$. Surplus is maximized by assigning players with lower indexes to positions with lower indexes. Following Moulin (1992), the characteristic function

$$v(S) = \sum_{j=1}^{|S|} \alpha_j y_j^{(S)}$$

defines the cooperative game.

Proposition 1 characterizes Shapley values for the position allocation problem.

Proposition 1: *The Shapley value ϕ_i of player i in the position allocation problem is*

$$\phi_i = \frac{1}{i} \left(\sum_{m=1}^{i} \alpha_m \right) x_i - \sum_{m=1}^{N-i} \frac{1}{i+m-1} \left[\sum_{r=1}^{i+m-1} \frac{r}{i+m} \left(\alpha_r - \alpha_{r+1} \right) x_{i+m} \right].$$

Following the interpretation of the Shapley value as the expected marginal contribution of a player, the first term in the expression for ϕ_i is the expected gross contribution of player i, while the second term captures the expected negative externality that i imposes on the other players.

Example 1 provides the Shapley values for the $N = 3$ problem.

Example 1: Suppose $N = 3$ and $x_1 > x_2 > x_3$. The players' Shapley values are:

$$
\begin{aligned}
\phi_1 &= \alpha_1 x_1 - \frac{1}{2}(\alpha_1 - \alpha_2)\, x_2 - \frac{1}{6}(\alpha_1 - \alpha_2)\, x_3 - \frac{1}{3}(\alpha_2 - \alpha_3)\, x_3, \\
\phi_2 &= \frac{1}{2}(\alpha_1 + \alpha_2)\, x_2 - \frac{1}{6}(\alpha_1 - \alpha_2)\, x_3 - \frac{1}{3}(\alpha_2 - \alpha_3)\, x_3, \\
\phi_3 &= \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)\, x_3.
\end{aligned}
$$

If $\alpha_1 = 6$, $\alpha_2 = 4$, and $\alpha_1 = 2$, and $x_1 = 3/4$, $x_2 = 1/2$, and $x_3 = 1/4$, then $\phi_1 = 15/4$, $\phi_2 = 9/4$, and $\phi_3 = 1$. In the Shapley allocation, player i receives position i. Players 1, 2, and 3, receive transfers of $-3/4$, $1/4$, and $1/2$, respectively.

The next section introduces a non-cooperative game for allocating positions. We will continue to develop Example 1 to illustrate our results.

3 The Compensated Position Auction

Here we propose an auction for solving the position allocation problem when the bidders' values are private information, and we characterize its equilibrium. The bidders' values are independently and identically distributed according to cumulative distribution function F with support $[0, \bar{x}]$, where $\bar{x} < \infty$ and $f \equiv F'$ is continuous and positive on $[0, \bar{x}]$. Bidders have a common utility function u, where $u' > 0$ and $u'' \leq 0$.

Let $X_1, \ldots, X_N$ be N independent draws from F. Let $Z_1^{(N)}, \ldots, Z_N^{(N)}$ be a rearrangement of the X_i's such that $Z_1^{(N)} \leq Z_2^{(N)} \leq \ldots \leq Z_N^{(N)}$. The joint density of $Z_1^{(N)}, \ldots, Z_N^{(N)}$ is

$$g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = N! \prod_{i=1}^{N} f(z_i)$$

for $z_1 \leq z_2 \leq \ldots \leq z_N$ and $g_{1,\ldots,N}^{(N)}(z_1, \ldots, z_N) = 0$ otherwise. Let $G_t^{(N)}$ denote the *c.d.f.* of $Z_t^{(N)}$, i.e., $G_t^{(N)}$ is the distribution of the t-th lowest of N draws. The conditional density of $Z_{t+1}^{(N)}$ given $Z_1^{(N)} = z_1, \ldots, Z_t^{(N)} = z_t$ is

$$g_{t+1}^{(N)}(z_{t+1}|z_t) = (N-t) f(z_{t+1}) \frac{[1 - F(z_{t+1})]^{N-(t+1)}}{[1 - F(z_t)]^{N-t}}$$

if $0 \leq z_1 \leq \ldots \leq z_{t+1}$ and is zero otherwise. Define

$$\lambda_t^N(z) \equiv g_{t+1}^{(N)}(z|z) = (N-t) \frac{f(z)}{1 - F(z)}$$

to be the hazard function.

The Auction

The auction takes place over $N-1$ rounds, where at each round the least desirable remaining position, i.e., the remaining position with the highest index, is allocated. At each round t, the price starts at zero and rises continuously. A bidder may drop out at any point. A bidder who drops out at price p_t is allocated position $N-t+1$ and he receives compensation of $p_t/(N-t)$ from each of the remaining $N-t$ bidders. Hence, the payoff of a bidder with value x who drops at round t at price p_t is

$$u\left(\alpha_{N-t+1} x + p_t - \sum_{s=1}^{t-1} \frac{p_s}{N-s}\right),$$

where

$$\sum_{s=1}^{t-1} \frac{p_s}{N-s}$$

is the compensation he pays to bidders who dropped at prior rounds. After $N-1$ bidders have dropped, the remaining bidder is allocated position 1, he receives no compensation, and for each $s \leq N-1$ he pays compensation $p_s/(N-s)$. His payoff when his value is x is

$$u\left(\alpha_1 x - \sum_{s=1}^{N-1} \frac{p_s}{N-s}\right).$$

In sum, a bidder who drops out surrenders his claim to more desirable positions and receives compensation from the bidders who maintain their claims to these positions, while he pays compensation to bidders who have accepted less desirable positions.

A strategy is a list of $N-1$ functions $\beta = (\beta_1, ..., \beta_{N-1})$, where $\beta_t(x; p_1, ..., p_{t-1})$ gives the dropout price in the t-th round of a bidder whose value is x, when $t-1$ bidders have previously dropped out at prices $p_1, \ldots, p_{t-1}$. We write $\mathbf{p}_{t-1}$ for $(p_1, \ldots, p_{t-1})$.

4 Equilibrium

Necessary Conditions for Equilibrium

Proposition 2 provides necessary conditions for β to be a symmetric equilibrium in strictly increasing and differentiable bidding strategies. These conditions are also sufficient for risk neutral and CARA bidders, as we establish in Propositions 3 and 4.

Proposition 2: *Any symmetric equilibrium β in increasing and differentiable bidding strategies satisfies the following system of $N-1$ differential equations:*

$$\begin{aligned} & u'\left(\alpha_{N-t+1}x + \beta_t(x;\mathbf{p}_{t-1}) - \sum\nolimits_{j=1}^{t-1} \frac{p_j}{N-j}\right)\beta_t'(x;\mathbf{p}_{t-1}) \\ = & -\left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta_{t+1}(x;\mathbf{p}_{t-1},\beta_t(x;\mathbf{p}_{t-1})) - \frac{1}{N-t}\beta_t(x;\mathbf{p}_{t-1}) - \sum_{j=1}^{t-1}\frac{p_j}{N-j}\right) \\ -u\left(\alpha_{N-t+1}x + \beta_t(x;\mathbf{p}_{t-1}) - \sum_{j=1}^{t-1}\frac{p_j}{N-j}\right) \end{array}\right]\lambda_t^N(x), \end{aligned}$$

for each $t \in \{1, \ldots, N-1\}$ where $\beta_N(x; \mathbf{p}_{N-1}) \equiv 0$.

Risk Neutral Bidders

Proposition 3 identifies the equilibrium when bidders are risk neutral. We write β_t^0 for the equilibrium bid function.

Proposition 3: *Suppose that bidders are risk neutral. The unique symmetric equilibrium in increasing and differentiable strategies is given, for* $t = 1, \ldots, N-1$, *by*

$$\beta_t^0(x) = \frac{N-t}{N-t+1} E\left[\left(\alpha_{N-t} - \alpha_{N-t+1}\right) Z_t^{(N)} + \beta_{t+1}^0(Z_t^{(N)}) | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]$$

where $\beta_N^0 \equiv 0$. *Equivalently, it is given by*

$$\beta_t^0(x) = \sum_{m=1}^{N-t} \frac{m}{N-t+1} E\left[\left(\alpha_m - \alpha_{m+1}\right) Z_{N-m}^{(N)} | Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].$$

Equilibrium bids at each round are independent of prior dropout prices.

Observe from the second expression for β_t^0 that if at some round t all the remaining positions have the same α's, i.e., $\alpha_1 = \ldots = \alpha_{N-t+1}$, then bids are zero at round t and every subsequent round. This is intuitive since when the remaining positions are identical and the number of positions is equal to the number of remaining bidders, then no position is contested.

Example 2: If $N = 3$ and values are distributed $U[0,1]$, then the equilibrium bid functions for risk neutral bidders are

$$\beta_1^0(x) = (\alpha_1 - \alpha_2)\left(\frac{1}{6}x + \frac{1}{6}\right) + (\alpha_2 - \alpha_3)\left(\frac{1}{2}x + \frac{1}{6}\right)$$

and

$$\beta_2^0(x) = (\alpha_1 - \alpha_2)\left(\frac{1}{3}x + \frac{1}{6}\right).$$

CARA Bidders

The next proposition characterizes equilibrium when bidders have constant absolute risk aversion (CARA), i.e., utility is given by

$$u^\theta(x) = \frac{1 - e^{-\theta x}}{\theta},$$

where $\theta > 0$ is the common index of risk aversion. Note that $\lim_{\theta \to 0} u^{\theta}(x) = x$, i.e., bidders are risk neutral in the limit as θ approaches zero. Denote by β_t^{θ} the equilibrium bid function in round t when bidders have CARA index of risk aversion θ.

Proposition 4: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$. The unique symmetric equilibrium in increasing and differentiable strategies is given recursively, for $t = 1, \ldots, N-1$, by*

$$\beta_t^{\theta}(x) = -\frac{N-t}{(N-t+1)\,\theta} \ln \left\{ E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)} + \beta_{t+1}^{\theta}(Z_t^{(N)})]} | Z_t^{(N)} > x > Z_{t-1}^{(N)} \right] \right\}$$

where $\beta_N^{\theta} \equiv 0$. Equilibrium bids at each round are independent of prior dropout prices.

Example 3: If $N = 3$ and values are distributed $U[0,1]$, then the equilibrium bid functions for CARA risk averse bidders are

$$\beta_1^{\theta}(x) = -\frac{2}{3\theta} \ln \left\{ \frac{\int_x^1 e^{-\theta\left[(\alpha_2-\alpha_3)z + \beta_2^{\theta}(z)\right]} 3(1-z)^2 dz}{(1-x)^3} \right\}$$

and

$$\beta_2^{\theta}(x) = -\frac{1}{2\theta} \ln \left\{ \frac{\int_x^1 e^{-\theta(\alpha_1-\alpha_2)z} 2(1-z) dz}{(1-x)^2} \right\}.$$

Bounds and Comparative Statics

Proposition 5 provides upper and lower bounds for the CARA equilibrium bid functions. The risk neutral bid function β_t^0 is an upper bound for the equilibrium bid function of a CARA risk averse bidder: a risk averse bidder drops out earlier, and thus accepts less compensation than a risk neutral bidder. The lower bound $\underline{\beta}_t$, defined below, will be central in the next section of the paper.

Proposition 5: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta > 0$ and $\alpha_1 > \alpha_2$. Then for each $t = 1, \ldots, N-1$ we have that*

$$\underline{\beta}_t(x) < \beta_t^\theta(x) < \beta_t^0(x) \text{ for } x < \bar{x},$$

where

$$\underline{\beta}_t(x) = \sum_{m=1}^{N-t} \frac{m}{N-t+1} (\alpha_m - \alpha_{m+1})\, x.$$

The lower bound $\underline{\beta}$ has a natural fairness interpretation akin to the solution to the contested garment problem as it calls for a bidder to demand equal shares of the incremental benefits obtained by the bidders allocated contested positions. At round 1, for example, positions 1 though $N-1$ are contested. (Position N is uncontested as it is the worst position.) There are $N-1$ bidders who will be allocated position $N-1$ *or better* and who will each enjoy an incremental benefit of $\alpha_{N-1} - \alpha_N$ (see row $N-1$ of Table 1). A bidder i with value x demands an equal share, $1/N$-th, of this total benefit as he values it, i.e., he demands $\frac{N-1}{N}(\alpha_{N-1} - \alpha_N)\, x$. There are $N-2$ bidders who will obtain position $N-2$ or better and who will each enjoy an incremental benefit of $\alpha_{N-2} - \alpha_{N-1}$. Bidder i demands an equal share of this total benefit too, i.e., $\frac{N-2}{N}(\alpha_{N-2} - \alpha_{N-1})\, x$. Continuing in this fashion, one bidder will obtain position 1 and enjoy an incremental benefit of $\alpha_1 - \alpha_2$. Bidder i demands an equal share. Adding up these shares of incremental benefits for contested positions yields $\underline{\beta}_1(x)$, bidder i's demand for compensation at round 1, as

$$\frac{1}{N}(\alpha_1 - \alpha_2)\, x + \cdots + \frac{N-2}{N}(\alpha_{N-2} - \alpha_{N-1})\, x + \frac{N-1}{N}(\alpha_{N-1} - \alpha_N)\, x.$$

The lower bound $\underline{\beta}_t$ has an interpretation analogous to $\underline{\beta}_1$, where equal shares are relative to the $N-t+1$ bidders remaining in the auction.

The risk neutral bid function β_t^0, given in Proposition 3, has a similar form and interpretation to $\underline{\beta}_t$. At round $t > 1$, positions are $1, \ldots, N-t$ are contested. The m-th term in $\underline{\beta}_t$, for $m \in \{1, \ldots, N-t\}$, i.e.,

$$\frac{m}{N-t+1}(\alpha_m - \alpha_{m+1})\, x,$$

is an equal share (among the $N-t+1$ active bidders at round t) of the total benefit obtained by the m bidders allocated position m or better, as a bidder with value x values it. The m-th term in β_t^0,

$$\frac{m}{N-t+1}\left(\alpha_m-\alpha_{m+1}\right)E\left[Z_{N-m}^{(N)}|Z_t^{(N)}>x>Z_{t-1}^{(N)}\right],$$

is the same except that bids are based on values that are inflated relative to the bidder's true value.[6]

Example 4 gives $\underline{\beta}_1(x)$ and $\underline{\beta}_2(x)$ when there are three positions.

Example 4: If $N=3$ then

$$\begin{aligned}\underline{\beta}_1(x) &= \frac{1}{3}\left(\alpha_1-\alpha_2\right)x+\frac{2}{3}\left(\alpha_2-\alpha_3\right)x\\ \underline{\beta}_2(x) &= \frac{1}{2}\left(\alpha_1-\alpha_2\right)x.\end{aligned}$$

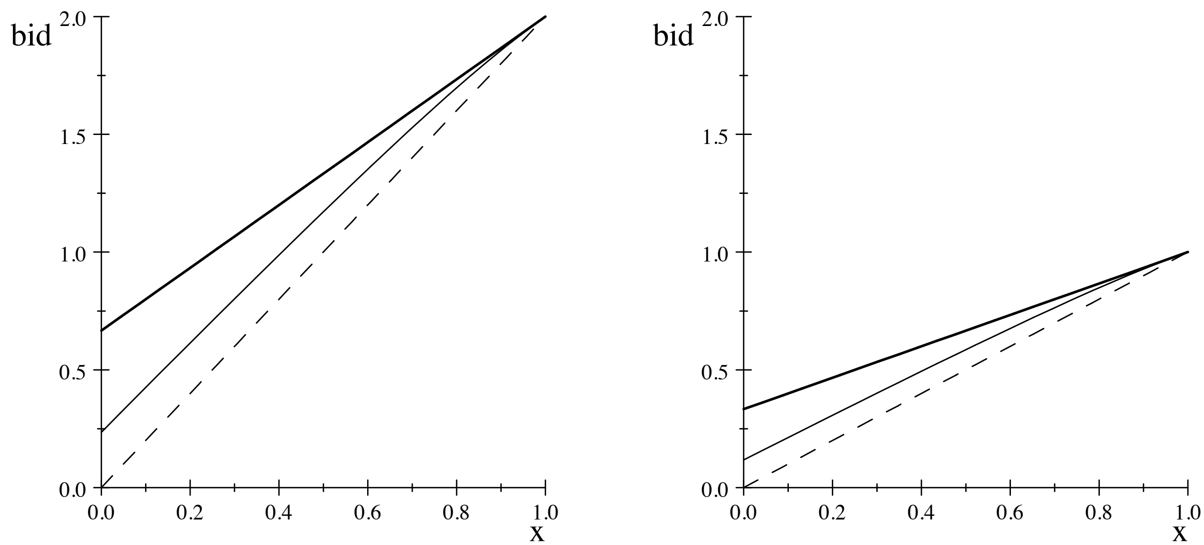
Proposition 6 shows that dropout prices decrease as bidders become more risk averse and that the lower bound provided in Proposition 5 is tight.

Proposition 6: *Suppose that bidders are CARA risk averse with index of risk aversion $\theta>0$. Then for each $t=1,\ldots,N-1$ we have that $\beta_t^\theta(x)$ is decreasing in θ, and β_t^θ converges uniformly to $\underline{\beta}_t$ on $[0,\bar{x}]$ as $\theta\to\infty$.*

Figure 1 illustrates propositions 5 and 6 when $N=3$, values are distributed $U[0,1]$, and $\alpha_1=6$, $\alpha_2=4$, and $\alpha_3=2$. In the figure, the bold solid lines are the risk neutral bid functions (i.e., $\theta=0$) for rounds 1 and 2, which are the upper bounds for CARA risk averse bidders. The dashed lines give $\underline{\beta}_1$ and $\underline{\beta}_2$, which are the lower bounds. The thin solid lines are the bid

[6] We have $Z_{N-m}^{(N)}\geq Z_t^{(N)}$ since $m\leq N-t$.

functions when bidders have CARA index of risk aversion of $\theta = 10$.



Round 1: $\theta = 0$, 10, and ∞.

Round 2: $\theta = 0$, 10, and ∞.

Figure 1: CARA Bounds

5 Maxmin

For a bidder who remains in the auction at round t (i.e., an "active" bidder), let $v_t(x_i, x_{-i}, \beta^i, \beta^{-i}; \mathbf{p}_{t-1})$ be the bidder's payoff when his value is x_i and he follows the strategy β^i, and x_{-i} and β^{-i} are the values and strategies of the remaining bidders, and $\mathbf{p}_{t-1}$ is the sequence of prior dropout prices.

Definition: A strategy β^i *guarantees bidder i with value x_i a payoff of $\bar{v}_t$ at round t, given $\mathbf{p}_{t-1}$*, if $v_t(x_i, x_{-i}, \beta^i, \beta^{-i}; \mathbf{p}_{t-1}) \geq \bar{v}_t \; \forall x_{-i}, \beta^{-i}$.

Let $\bar{v}_t(x_i; \mathbf{p}_{t-1})$ be the largest payoff that bidder i with value x_i can guarantee at round t given $\mathbf{p}_{t-1}$.

Definition: A strategy β^i is a *maxmin perfect strategy for bidder i* if β^i guarantees $\bar{v}_t(x_i; \mathbf{p}_{t-1})$ for each t, $x_i \in [0, \bar{x}]$, and $\mathbf{p}_{t-1}$.

Proposition 7: *The bidding strategy $\underline{\beta} = (\underline{\beta}_1, \ldots, \underline{\beta}_{N-1})$ given in Proposition 5 is the unique maxmin perfect strategy of the compensated position auction. In particular, $\underline{\beta}$ guarantees a bidder with value x a payoff at round t of*

$$\bar{v}_t(x; \mathbf{p}_{t-1}) = \left(\sum_{m=1}^{N-t+1} \frac{\alpha_m}{N-t+1} \right) x - \sum_{i=1}^{t-1} \frac{p_i}{N-i},$$

when $\mathbf{p}_{t-1}$ is the sequence of dropout prices.

An immediate implication of Proposition 7 is that participation in the auction is individually rational. Following his maxmin perfect strategy, a bidder with value x guarantees himself a payoff of at least $\frac{1}{N}\sum_{m=1}^{N} \alpha_m x$ and a utility of at least $u(\frac{1}{N}\sum_{m=1}^{N} \alpha_m x)$. His equilibrium expected utility is therefore at least $u(\frac{1}{N}\sum_{m=1}^{N} \alpha_m x)$. Concavity of u implies that

$$u(\frac{1}{N}\sum_{m=1}^{N} \alpha_m x) \geq \frac{1}{N}\sum_{m=1}^{N} u(\alpha_m x),$$

and thus bidders would rather participate in the auction than not, when the alternative is the random allocation of positions.

6 Equilibrium, Maxmin, and the Shapley Value

Proposition 8 provides the decision theoretic foundation of the Shapley value in the compensated position auction.

Proposition 8: *If each bidder follows his maxmin perfect strategy then each bidder obtains his Shapley value allocation.*

Example 5 illustrates Proposition 8 when $N = 3$.

Example 5: Suppose, as in Example 1 that $x_1 > x_2 > x_3$. If each bidder follows his maxmin perfect strategy $\underline{\beta}$, given in Example 4, then Bidder 3 drops first at $\underline{\beta}_1(x_3)$, he wins position three, and his payoff is

$$\alpha_3 x_3 + \underline{\beta}_1(x_3) = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3) x_3 = \phi_3.$$

Bidder 2 drops second at $\underline{\beta}_2(x_2)$, he wins position two, he receives compensation $\underline{\beta}_2(x_2)$ and he pays compensation $\frac{1}{2}\underline{\beta}_1(x_3)$. His payoff is

$$\begin{aligned} & \alpha_2 x_2 + \underline{\beta}_2(x_2) - \frac{1}{2}\underline{\beta}_1(x_3) \\ = & \frac{1}{2}(\alpha_1+\alpha_2)\,x_2 - \frac{1}{6}(\alpha_1-\alpha_2)\,x_3 - \frac{1}{3}(\alpha_2-\alpha_3)\,x_3 \\ = & \phi_2. \end{aligned}$$

Bidder 1 wins position 1 and pays total compensation of $\underline{\beta}_2(x_2) + \frac{1}{2}\underline{\beta}_1(x_3)$. His payoff is

$$\begin{aligned} & \alpha_1 x_1 - \underline{\beta}_2(x_2) - \frac{1}{2}\underline{\beta}_1(x_3) \\ = & \alpha_1 x_1 - \frac{1}{2}(\alpha_1-\alpha_2)\,x_2 - \frac{1}{6}(\alpha_1-\alpha_2)\,x_3 - \frac{1}{3}(\alpha_2-\alpha_3)\,x_3 \\ = & \phi_1. \end{aligned}$$

Thus, each bidder receives his Shapley value.

By Proposition 6, as bidders become infinitely risk averse, the equilibrium bid function converges to the maxmin perfect bid function $\underline{\beta}$. By Proposition 8, when each bidder follows his maxmin perfect strategy, then each obtains his Shapley value allocation. Combining these two results yields the following Corollary.

Corollary 1: *As bidders become infinitely risk averse, the equilibrium allocation approaches the Shapley-value allocation.*

The next example and the associated figure illustrate Corollary 1, showing that the bidders' realized payoffs converge to their Shapley value payoffs as bidders become infinitely risk averse.

Example 6: Figure 2 shows the equilibrium payoff of each bidder as a function of θ, when $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_3 = 2$ and the bidders' values are $x_1 = 3/4$, $x_2 = 1/2$, and $x_3 = 1/4$. The payoff of bidder 3 is

$$y_3(\theta) := \alpha_3 x_3 + \beta_1^{\theta}(x_3),$$

of bidder 2 is

$$y_2(\theta) := \alpha_2 x_2 + \beta_2^\theta(x_2; \beta_1^\theta(x_3)) - \frac{1}{2}\beta_1^\theta(x_3),$$

of bidder 1 is

$$y_1(\theta) := \alpha_1 x_1 - \beta_2^\theta(x_2; \beta_1^\theta(x_3)) - \frac{1}{2}\beta_1^\theta(x_3).$$

The dashed lines are the bidders' Shapley values.

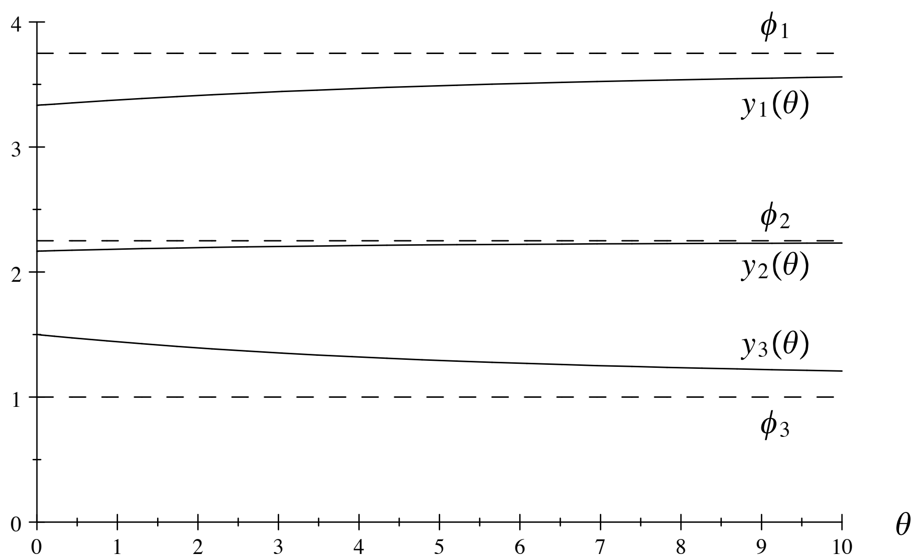


Figure 2: Equilibrium Payoffs as a Function of θ.

Since the auction is efficient, each bidder is allocated the same position he would receive in the Shapley allocation. As θ approaches infinity, each bidder also receives the same transfer that he would receive in the Shapley allocation: Bidder 3 receives compensation of $\underline{\beta}_1(1/4) = 1/2$. Bidder 2 receives compensation of $\underline{\beta}_2(1/2) = 1/2$ from Bidder 1, but pays compensation of $\frac{1}{2}\underline{\beta}_1(1/4)$ to Bidder 3, for a net transfer of $1/4$. Bidder 1 pays compensation of $\frac{1}{2}\underline{\beta}_1(1/4)$ to Bidder 3 and $\underline{\beta}_2(1/2)$ to Bidder 2, for a net transfer of $-3/4$. These are exactly the transfers identified in Example 1.

7 Discussion

This paper proposes a solution to the problem of fairly allocating heterogeneous items, priorities, positions, or rights among participants who have equal claims. The auction we propose is efficient and budget balanced. From a purely theoretical perspective it is of interest since it provides non-cooperative and decision theoretic foundations for the Shapley value in an environment with incomplete information.

While we have framed the compensated position auction as multi-round ascending bid auction, it is strategically equivalent to the multi-round sealed bid auction in which, at each round, the bidders simultaneously make bids (i.e., demand compensation), and the bidder with the lowest bid is allocated the worst remaining position and receives his bid as compensation. The reasoning is the same as for the strategic equivalence of the Dutch and first price sealed bid auction.

There may be other auctions whose Bayes Nash equilibria converge to the Shapley value as bidders become infinitely risk averse and which generate Shapley value allocations under maxmin play. It is easy, however, to construct auctions that do not have these properties. Consider, for example, the auction in which all bidders simultaneously make sealed bids, the highest bidder gets the best position, the second highest bidder gets the second best position, and so on. In the auction, only the highest bidder pays his bid and his bid is divided equally among all the bidders. If the auction has a symmetric equilibrium in increasing strategies, then the auction will be efficient and budget balanced. It cannot, however, generate the Shapley allocation as all the bidders (except the highest) receive the same net transfer, namely $1/N$-th of the highest bid. As Example 1 illustrates, the Shapley allocation requires different bidders receive different net transfers.

The cooperative game we study admits a non-empty *anti-core* and the Shapley value is a member of this set. Given a characteristic function v, a payoff vector $(\pi_1, \ldots, \pi_N)$ is in the anti-core if (i) $\sum_{i \in N} \pi_i = v(N)$ and (ii) for every coalition $S \subset N$ we have that $\sum_{i \in S} \pi_i \leq v(S)$. In other words, a payoff vector is in the anti-core if no coalition of players receives more than what it could obtain if the coalition had complete command over the allocation of resources. A payoff to S that exceeds $v(S)$ requires a subsidy from $N\backslash S$, which would object on fairness grounds. (Observe that the anti-core of a cooperative game is motivated by normative/fairness considerations, in contrast to the core which is motivated by strategic considerations. See Moulin (1995, Chapter 7).) Moulin (1992, Theorem 2) established that the general assignment game is concave, and thus the problem of assigning players to positions is also concave. It follows, by Shapley (1971, Theorems 4 and 7), that the anti-core of the game is non-empty and contains the Shapley value. These results imply that the compensated position auction produces allocations in the anti-core when bidders are sufficiently risk averse or when each bidder follows the maxmin perfect strategy.

8 Appendix

The proof of Proposition 1 involves combinatorial arguments that play no role in the remaining proofs. It is included for completeness, but the reader is invited to skip it.

Proof of Proposition 1: We compute the Shapley value directly using that

$$\phi_i = \sum_{s=1}^{N} \frac{(N-s)!(s-1)!}{N!} \left[\sum_{B_i(s)} (v(S) - v(S\backslash\{i\})) \right]$$

where

$$B_i(s) = \{S | i \in S \text{ and } |S| = s\}.$$

We first compute the marginal contribution of player i to coalition S. If

$i \in S$ has the j-th highest value in coalition S (i.e., $x_i = y_j^{(S)}$) then

$$
\begin{aligned}
v(S) - v(S \backslash \{i\}) &= \alpha_j y_j^{(S)} - \sum_{m=1}^{|S|-j} (\alpha_{j+m-1} - \alpha_{j+m})\, y_{j+m}^{(S)} \\
&= \alpha_j x_i - \sum_{m=1}^{|S|-j} (\alpha_{j+m-1} - \alpha_{j+m})\, y_{j+m}^{(S)}.
\end{aligned}
$$

This follows since in coalition S player i is assigned the j-th position, players in S with a smaller index than i stay in the same position they occupied in $S \backslash \{i\}$, and players with a higher index than i move down one position.

Player i's Shapley value can be written as

$$\phi_i = c^i x_i - \sum_{m=1}^{N-i} d^{im} x_{i+m},$$

where c^i is of the form

$$c^i = c_1^i \alpha_1 + \cdots + c_i^i \alpha_i,$$

and d^{im} is of the form

$$d^{im} = d_1^{im} (\alpha_1 - \alpha_2) + \cdots + d_{i+m-1}^{im} (\alpha_{i+m-1} - \alpha_{i+m}).$$

The term c^i is the expected contribution of player i and $d^{im} x_{i+m}$ is the expected externality that i imposes on player $i+m$.

We now compute c_r^i for $1 \leq r \leq i$, which is the contribution of player i when allocated position r. For each coalition size s, we count the number of coalitions of size s where i is in position r and multiply this number by the appropriate Shapley weight. The coefficient c_r^i is the sum of these terms over all s.

The smallest coalitions where i is in position r are coalitions of size r, and consist of player i and $r-1$ players with a smaller index. The largest coalitions where i is in position r are coalitions of size $N - i + r$, and consist of player i, $r-1$ players with a smaller index, and $N-i$ players with a larger index. The number of coalitions of size s where i is placed in position r is

$$\binom{i-1}{r-1}\binom{N-i}{s-r},$$

where $\binom{i-1}{r-1}$ is the number of ways of choosing $r-1$ players with index smaller than i from $i-1$ players, and $\binom{N-i}{s-r}$ is the number of ways of choosing $s-r$ players with index larger than i from $N-i$ players. The Shapley weight for coalitions of size s is

$$\frac{(s-1)!(N-s)!}{N!},$$

and therefore

$$c_r^i = \sum_{s=r}^{N-i+r} \frac{(s-1)!(N-s)!}{N!}\binom{i-1}{r-1}\binom{N-i}{s-r}.$$

Summing across positions where player i can be placed yields

$$\begin{aligned} c^i &= \sum_{r=1}^{i}\left[\sum_{s=r}^{N-i+r} \frac{(s-1)!(N-s)!}{N!}\binom{i-1}{r-1}\binom{N-i}{s-r}\right]\alpha_r \\ &= \sum_{r=1}^{i}\left[\frac{1}{N}\sum_{s=r}^{N-i+r}\frac{\binom{i-1}{r-1}\binom{N-i}{s-r}}{\binom{N-1}{s-1}}\right]\alpha_r \\ &= \frac{1}{i}\sum_{r=1}^{i}\alpha_r, \end{aligned}$$

where the last equality holds by Claim 4 in the Supplemental Appendix.

Next, we compute d_r^{im} for $0 < m \leq N-i$ and $1 \leq r < i+m$. The term $d_r^{im}(\alpha_r - \alpha_{r+1})x_{i+m}$ will be the expected externality player i imposes on player $i+m$ by pushing player $i+m$ from r to $r+1$. For each player $i+m$, position r, and coalition size s, we count the number of coalitions of size s where player i pushes player $i+m$ from position r to position $r+1$ and we multiply this number by the appropriate Shapley weight. The coefficient d_r^{im} is the sum of these terms over all s.

The smallest coalitions where i pushes $i+m$ from position r to position $r+1$ are coalitions of size $r+1$, and consist of player i, player $i+m$, and $r-1$ other players with smaller index than $i+m$. The largest coalitions where i pushes $i+m$ from position r to position $r+1$ are coalitions of size $r+1+N-(i+m)$, and consist of player i, player $i+m$, $r-1$ other players with index smaller than $i+m$, and the $N-(i+m)$ players with an index larger than $i+m$. The number of coalitions of size s where i pushes $i+m$ from position r to position $r+1$ is

$$\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)},$$

where $\binom{i+m-2}{r-1}$ is the number of ways of choosing $r-1$ players (excluding player i) with index smaller than $i+m$, and $\binom{N-(i+m)}{s-(r+1)}$ is the number of ways of choosing $s-(r+1)$ players with index larger than $i+m$ from $N-(i+m)$ players. The Shapley weight for coalitions of size s is

$$\frac{(s-1)!(N-s)!}{N!},$$

and therefore,

$$d_r^{im} = \sum_{s=r+1}^{r+1+N-(i+m)} \frac{(s-1)!(N-s)!}{N!}\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}.$$

Summing across positions where player $i+m$ can be placed yields

$$\begin{aligned} d^{im} &= \sum_{r=1}^{i+m-1}\left[\sum_{s=r+1}^{r+1+N-(i+m)} \frac{(s-1)!(N-s)!}{N!}\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}\right](\alpha_r-\alpha_{r+1}) \\ &= \sum_{r=1}^{i+m-1}\left[\frac{1}{N}\sum_{s=r+1}^{r+1+N-(i+m)} \frac{\binom{i+m-2}{r-1}\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{S-1}}\right](\alpha_r-\alpha_{r+1}). \end{aligned}$$

The identity in Claim 4 holds for all $i \leq N$. Replacing i with $i+m$ and r with $r+1$ in this identity, and noting that $i+m \leq N$ also, we obtain the following new identity

$$\frac{1}{N}\sum_{s=(r+1)}^{N+(r+1)-(i+m)}\binom{(i+m)-1}{(r+1)-1}\frac{\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{s-1}}=\frac{1}{i+m}.$$

Applying this new identity to d^{im} yields

$$\begin{aligned}
d^{im} &= \sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{(i+m)-1}{(r+1)-1}}\left[\frac{1}{N}\sum_{s=r+1}^{r+1+N-(i+m)}\frac{\binom{(i+m)-1}{(r+1)-1}\binom{N-(i+m)}{s-(r+1)}}{\binom{N-1}{S-1}}\right](\alpha_r-\alpha_{r+1}) \\
&= \frac{1}{i+m}\sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}}(\alpha_r-\alpha_{r+1}).
\end{aligned}$$

The total expected externality that player i imposes on the other players is

$$\begin{aligned}
\sum_{m=1}^{N-i} d^{im}x_{i+m} &= \sum_{m=1}^{N-i}\left[\frac{1}{i+m}\sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}}(\alpha_r-\alpha_{r+1})\right]x_{i+m} \\
&= \sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\frac{i+m-1}{i+m}\sum_{r=1}^{i+m-1}\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}}(\alpha_r-\alpha_{r+1})\right]x_{i+m}.
\end{aligned}$$

Noting that

$$\begin{aligned}
(i+m-1)\frac{\binom{i+m-2}{r-1}}{\binom{i+m-1}{r}} &= (i+m-1)\frac{\frac{(i+m-2)!}{(i+m-2-(r-1))!(r-1)!}}{\frac{(i+m-1)!}{(i+m-1-r))!r!}} \\
&= (i+m-1)\frac{\frac{(i+m-2)!}{(i+m-1-r)!(r-1)!}}{\frac{(i+m-1)!}{(i+m-1-r))!r!}} \\
&= r,
\end{aligned}$$

we can write

$$\sum_{m=1}^{N-i} d^{im}x_{i+m}=\sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1}\frac{r}{i+m}(\alpha_r-\alpha_{r+1})\,x_{i+m}\right].$$

Collecting terms, the Shapley value of player i is

$$\phi_i = \frac{1}{i}\left(\sum_{m=1}^{i}\alpha_m\right)x_i - \sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1}\frac{r}{i+m}\left(\alpha_r-\alpha_{r+1}\right)x_{i+m}\right].$$

□

Proof of Proposition 2: Let $\beta = (\beta_1, \ldots, \beta_{N-1})$ be a symmetric equilibrium in increasing and differentiable strategies. Since equilibrium is in increasing strategies, the sequence of dropout prices $(p_1, \ldots, p_{t-1})$ at round t reveals the $t-1$ lowest values $(z_1, \ldots, z_{t-1})$. In the proof it is convenient to write the round t equilibrium bid as a function of the prior dropout values rather than as a function of the prior dropout prices. In particular, we write $\beta_t(x|\mathbf{z}_{t-1})$ rather than $\beta_t(x;\mathbf{p}_{t-1})$.

For each $t < N$, let $\pi_t(y, x|\mathbf{z}_{t-1})$ be the expected payoff to a bidder with value x who in round t deviates from equilibrium and bids as though his value is y (i.e., he bids $\beta_t(y|\mathbf{z}_{t-1})$), when $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop so far. In this case we will sometimes say the bidder "bids y". Let

$$\Pi_t(x|\mathbf{z}_{t-1}) = \pi_t(x, x|\mathbf{z}_{t-1})$$

be the equilibrium payoff of a bidder in round t when his value is x and $\mathbf{z}_{t-1}$ is the profile of values of the $t-1$ bidders to drop in prior rounds.

We now derive the necessary conditions in Proposition 2. Let $\mathbf{z}_{t-1}$ be arbitrary. Consider a bid y. If $z_t \in [z_{t-1}, y]$ the bidder continues to round $t+1$ and has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j$ in round t. Hence his expected payoff is

$$\begin{aligned}\pi_t(y, x|\mathbf{z}_{t-1}) = \; & \textstyle\int_{z_{t-1}}^{y}\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)g_t^{(N-1)}(z_t|z_{t-1})dz_t \\ & + \textstyle\int_{y}^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right) g_t^{(N-1)}(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial\pi_t(y, x|\mathbf{z}_{t-1})}{\partial y} = \; & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - u\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}p_j\right)]g_t^{(N-1)}(y|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}p_j\right)\beta_t'(y|\mathbf{z}_{t-1})(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

A necessary condition for equilibrium is that $\partial \pi_t(y, x|\mathbf{z}_{t-1})/\partial y|_{y=x} = 0$, i.e.,

$$[\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) - u\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}p_j\right)]g_t^{(N-1)}(x|z_{t-1})$$
$$+u'\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\tfrac{1}{N-j}p_j\right)\beta_t'(x|\mathbf{z}_{t-1})(1 - G_t^{(N-1)}(x|z_{t-1})) = 0.$$

Since

$$\begin{aligned}\Pi_{t+1}(x|\mathbf{z}_{t-1}, x) &= \pi_{t+1}(x, x|\mathbf{z}_{t-1}, x) \\ &= u\left(\alpha_{N-t}x + \beta_{t+1}(x|\mathbf{z}_{t-1}, x) - \frac{1}{N-t}\beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right)\end{aligned}$$

the necessary condition can be written as

$$\begin{aligned}& u'\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right)\beta_t'(x|\mathbf{z}_{t-1}) \\ = & -\left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta_{t+1}(x|\mathbf{z}_{t-1}, x) - \frac{1}{N-t}\beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right) \\ -u\left(\alpha_{N-t+1}x + \beta_t(x|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right)\end{array}\right]\lambda_t^N(x),\end{aligned}$$

where $\beta_N(x; \mathbf{z}_{N-1}) \equiv 0$. Replacing $\mathbf{z}_{t-1}$ with $\mathbf{p}_{t-1}$ and the x in $\beta_{t+1}(x|\mathbf{z}_{t-1}, x)$ with $\beta_t(x|\mathbf{p}_{t-1})$ yields the differential equation given in the Proposition for round t. $\square$

Proof of Proposition 3: We first show that the bidding functions in Proposition 3 satisfies the system of differential equations in Proposition 2. The proof is by induction. Consider round $N-1$. The differential equation for round $N-1$ is

$$\beta_{N-1}^{0\prime}(x|\mathbf{z}_{N-2}) = -[(\alpha_1 - \alpha_2)x - 2\beta_{N-1}^0(x|\mathbf{z}_{N-2})]\lambda_{N-1}^N(x). \qquad (1)$$

The unique solution is

$$\begin{aligned}\beta_{N-1}^0(x) &= \frac{1}{2}\frac{\int_x^{\bar{x}}(\alpha_1 - \alpha_2)\, z2f(z)(1 - F(z))dz}{(1 - F(x))^2} \\ &= \frac{1}{2}E\left[(\alpha_1 - \alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right],\end{aligned}$$

which is $\beta_{N-1}^0(x)$, as given in Proposition 3.

Suppose $\beta^0_{t+1},\ldots,\beta^0_{N-1}$ are as given in Proposition 3 for round $t+1,\ldots,N-1$. Consider round t. The differential equation in Proposition 2 for round t, using the notation from the proof of Proposition 2, is

$$\beta^{0\prime}_t(x|\mathbf{z}_{t-1}) = -\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^0_{t+1}(x|\mathbf{z}_{t-1},x)-\frac{N-t+1}{N-t}\beta^0_t(x|\mathbf{z}_{t-1})\right]\lambda^N_t(x).$$

Since $\beta^0_{t+1}(x|\mathbf{z}_{t-1},x)$ is independent of $(\mathbf{z}_{t-1},x)$, we can write

$$\beta^{0\prime}_t(x) = -\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^0_{t+1}(x)-\frac{N-t+1}{N-t}\beta^0_t(x)\right]\lambda^N_t(x).$$

The unique solution is

$$\begin{aligned}
\beta^0_t(x) &= \frac{N-t}{N-t+1}\int_x^{\bar{x}}\frac{\left((\alpha_{N-t}-\alpha_{N-t+1})z+\beta^0_{t+1}(z)\right)(N-t+1)f(z)(1-F(z))^{N-t}}{(1-F(x))^{N-t+1}}dz\\
&= \frac{N-t}{N-t+1}E\left[\left(\alpha_{N-t}-\alpha_{N-t-1}\right)Z^{(N)}_t+\beta^0_{t+1}(Z^{(N)}_t)|Z^{(N)}_t>x>Z^{(N)}_{t-1}\right]\\
&= \sum_{m=1}^{N-t}\frac{m}{N-t+1}E\left[\left(\alpha_m-\alpha_{m+1}\right)Z^{(N)}_{N-m}|Z^{(N)}_t>x>Z^{(N)}_{t-1}\right],
\end{aligned}$$

where the second equality restates the first equality as an expected value. The third equality is established as Claim 5 in the Supplemental Appendix. This establishes the result for round t and hence, by induction, the result for all t.

Next we establish that the bidding strategies are an equilibrium. It is sufficient to show that the following three-part claim holds for every t:

1. If $x\geq z_{t-1}$ then $x\in\arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$, i.e., it is optimal for a bidder with value x to bid $\beta^0_t(x)$ in round t.

2. If $x<z_{t-1}$ then $z_{t-1}\in\arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$.

3. $\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx}\geq\alpha_{N-t+1}$.

Parts 2 and 3 are ancillary results needed to establish Part 1 for rounds prior to the last round. Part 2 is necessary to evaluate the consequence at round t of a bid y greater than the equilibrium bid x. In this case, a rival bidder with value $z_t > x$ may drop out before the bidder, and we need to evaluate the consequence for his optimal bid in round $t+1$. Part 2 shows that in this event it is optimal for the bidder to bid z_t (rather than x) in round $t+1$.

The proof is by induction. Consider round $N-1$. Any bid below z_{N-2} is strictly dominated by a bid of z_{N-2} since both bids result in the same position while a bid of z_{N-2} yields higher compensation. Suppose $y \geq z_{N-2}$. When bidders are risk neutral we have

$$\begin{aligned}\pi_{N-1}(y,x|\mathbf{z}_{N-2}) = & \int\limits_{z_{N-2}}^{y} \left(\alpha_1 x - \beta^0_{N-1}(z_{N-1}) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} p_j\right) g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2}) dz_{N-1} \\ & + \int\limits_{y}^{\bar{x}} \left(\alpha_2 x + \beta^0_{N-1}(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} p_j\right) g_{N-1}^{(N-1)}(z_{N-1}|z_{N-2}) dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{aligned} & (\alpha_1 x - \beta^0_{N-1}(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} p_j) g_{N-1}^{(N-1)}(y|z_{N-2}) \\ - \ & (\alpha_2 x + \beta^0_{N-1}(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} p_j) g_{N-1}^{(N-1)}(y|z_{N-2}) \\ + \ & \beta^{0\prime}_{N-1}(y)(1 - G_{N-1}^{(N-1)}(y|z_{N-2})).\end{aligned}$$

Substituting the differential equation (1)

$$\beta^{0\prime}_{N-1}(y) = -[(\alpha_1 - \alpha_2)y - 2\beta^0_{N-1}(y)]\lambda^N_{N-1}(y) \tag{2}$$

into the expression for $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y$ yields

$$\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y = (\alpha_1 - \alpha_2)(x-y) g_{N-1}^{(N-1)}(y|z_{N-2}).$$

If $y < x$ then $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y > 0$, and if $y > x$ then $\partial\pi_{N-1}(y,x|\mathbf{z}_{N-2})/\partial y < 0$. Thus $x \geq z_{N-2}$ implies $x \in \arg\max_y \pi_{N-1}(y,x|\mathbf{z}_{N-2})$, which establishes Part 1.

If $x < z_{N-2}$, then any bid below z_{N-2} is strictly dominated. For any bid $y \geq z_{N-2}$ then $y > x$ and the above argument establishes $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$ for all $y \geq z_{N-2}$, i.e., $z_{N-2} \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$. This establishes Part 2.

By the Envelope Theorem

$$\begin{aligned}\frac{d\Pi_{N-1}(x|\mathbf{z}_{N-2})}{dx} &= \left.\frac{\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})}{\partial x}\right|_{y=x} \\ &= \alpha_1 G_{N-1}^{(N-1)}(x|z_{N-2}) + \alpha_2(1 - G_{N-1}^{(N-1)}(x|z_{N-2})) \\ &\geq \alpha_2,\end{aligned}$$

which establishes Part 3. This completes the claim for round $N-1$.

Assume the three-part claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be a sequence of dropout values. Suppose $x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids y. A bid below z_{t-1} is dominated. Since his payoff function differs in each case, we need to distinguish (i) $y \in [z_{t-1}, x]$ and (ii) $y > x$. In what follows, we denote the payoff to a bid of y as $\pi_t^L(y, x|\mathbf{z}_{t-1})$ if $y \in [z_{t-1}, x]$ and as $\pi_t^H(y, x|\mathbf{z}_{t-1})$ if $y \geq x$.

Case (i): Consider a bid $y \in [z_{t-1}, x]$. If the next highest value of a rival bidder is $z_t \in [z_{t-1}, y]$, then the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j$ in round t. Hence his payoff is

$$\begin{aligned}\pi_t^L(y, x|\mathbf{z}_{t-1}) = \ & \textstyle\int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_{y}^{\bar{x}} \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial\pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial y} = \ & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j\right)] g_t^{(N-1)}(y|z_{t-1}) \\ & + \beta_t^{0\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

By the induction hypothesis we have

$$\frac{\partial^2 \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial x \partial y} = \left(\frac{d\Pi_{t+1}(x|\mathbf{z}_{t-1},y)}{dx} - \alpha_{N-t+1} \right) g_t^{(N-1)}(y|z_{t-1}) \geq 0.$$

Case (ii): Consider a bid $y \geq x$. If the next highest value of a rival bidder is $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by the induction hypothesis, he bids x and obtains $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, y]$, then he continues to round $t+1$ and, by the part 2 of the induction hypothesis, he optimally bids z_t, he wins position $N-t$, and obtains compensation $\beta_{t+1}^0(z_t)$. His payoff is $\alpha_{N-t}x+ \beta_{t+1}^0(z_t) - \frac{1}{N-t}\beta_t^0(z_t) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j$. If $z_t > y$, then in round t his payoff is $\alpha_{N-t+1}x+ \beta_t^0(y|\mathbf{z}_{t-1}) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j$. Thus his expected payoff at round t is

$$\begin{aligned}\pi_t^H(y,x|\mathbf{z}_{t-1}) = \ & \textstyle\int_{z_{t-1}}^{x} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_x^y \left(\alpha_{N-t}x + \beta_{t+1}^0(z_t) - \frac{1}{N-t}\beta_t^0(z_t) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j \right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_y^{\bar{x}} \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j \right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial y} \ = \ & \left[\left((\alpha_{N-t} - \alpha_{N-t+1})x + \beta_{t+1}^0(y) - \frac{N-t+1}{N-t}\beta_t^0(y) \right) \right] g_t^{(N-1)}(y|z_{t-1}) \\ & + \beta_t^{0\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Since $\alpha_{N-t} - \alpha_{N-t+1} \geq 0$ then

$$\frac{\partial^2 \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial x \partial y} = (\alpha_{N-t} - \alpha_{N-t+1}) g_t^{(N-1)}(y|z_{t-1}) \geq 0.$$

We have shown that

$$\left. \frac{\partial \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial y} \right|_{y=x} = \left. \frac{\partial \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} \right|_{y=x} = 0$$

and

$$\frac{\partial^2 \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial x \partial y} \geq 0 \text{ for } y \leq x \text{ and } \frac{\partial^2 \pi_t^H(y,x|\mathbf{z}_{t-1})}{\partial x \partial y} \geq 0 \text{ for } y \leq x.$$

hence by Lemma 0 in Van Essen and Wooders (2016) we have $x \in \arg\max_{y \in [z_{t-1}, \bar{x}]} \pi_t(y, x|\mathbf{z}_{t-1})$. This establishes Part 1 for round t.

Suppose $x < z_{t-1}$. Any $y < z_{t-1}$ is strictly dominated by a bid of z_{t-1}. For $y \geq z_{t-1}$ we can write

$$\begin{aligned}\pi_t(y, x|\mathbf{z}_{t-1}) = \; & \int_{z_{t-1}}^{y} \left(\alpha_{N-t}x + \beta_{t+1}^0(z_t) - \tfrac{1}{N-t}\beta_t^0(z_t) - \Sigma_{j=1}^{t-1} \tfrac{1}{N-j} p_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \int_{y}^{\bar{x}} \left(\alpha_{N-t+1}x + \beta_t^0(y) - \Sigma_{j=1}^{t-1} \tfrac{1}{N-j} p_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to y and replacing $\beta_t^{0'}(y)$ with the equilibrium differential equation yields

$$\frac{\partial \pi_t(y, x|\mathbf{z}_{t-1})}{\partial y} = (\alpha_{N-t} - \alpha_{N-t+1})(x - y)\, g_t^{(N-1)}(y|z_{t-1}) \leq 0$$

since $y > x$ and $\alpha_{N-t} - \alpha_{N-t+1} \geq 0$. Hence, if $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$. This establishes Part 2 for round t.

Finally, by the Envelope Theorem, we have

$$\begin{aligned}\frac{d\Pi_t(x|\mathbf{z}_{t-1})}{dx} &= \left.\frac{\partial \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial x}\right|_{y=x} = \left.\frac{\partial \pi_t^H(y, x|\mathbf{z}_{t-1})}{\partial x}\right|_{y=x} \\ &= \int_{z_{t-1}}^{x} \frac{d\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)}{dx} g_t^{(N-1)}(z_t|z_{t-1}) dz_t + \alpha_{N-t+1}(1 - G_t^{(N-1)}(x|z_{t-1}) \\ &\geq \alpha_{N-t} G_t^{(N-1)}(x|z_{t-1}) + \alpha_{N-t+1}(1 - G_t^{(N-1)}(x|z_{t-1}) \\ &\geq \alpha_{N-t+1}\end{aligned}$$

where the first inequality follows from the induction hypothesis and the second inequality follows since $\alpha_{N-t} \geq \alpha_{N-t+1}$. This establishes Part 3 for round t, and completes the proof by induction. $\square$

Proof of Proposition 4: We first show that the bidding functions given in Proposition 4 are the unique solution to the system of differential equations in Proposition 2 when bidders have CARA utility. The proof is by induction. Consider round $N-1$. The differential equation for round $N-1$ is

$$-\theta e^{-\theta[\alpha_2 x + \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j]} \beta_{N-1}^{\theta\prime}(x|\mathbf{z}_{N-2}) =$$
$$[e^{-\theta[\alpha_2 x + \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j]} - e^{-\theta[\alpha_1 x - \beta_{N-1}^{\theta}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j]}] \lambda_{N-1}^{N}(x).$$

Dividing both sides by $e^{-\theta[\alpha_2 x - \beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2}) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]}$ yields

$$-\theta e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) = [e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})} - e^{-\theta(\alpha_1-\alpha_2)x}]\lambda^{N}_{N-1}(x).$$

Multiplying both sides by $2(1-F(x))^2$ yields

$$\begin{aligned} & -2(1-F(x))^2\theta e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) \\ = \; & 2f(x)(1-F(x))[e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})} - e^{-\theta(\alpha_1-\alpha_2)x}]. \end{aligned}$$

Rearranging

$$\begin{aligned} & -2\theta(1-F(x))^2 e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}\beta^{\theta\prime}_{N-1}(x|\mathbf{z}_{N-2}) - 2f(x)(1-F(x))e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})} \\ = \; & -e^{-\theta(\alpha_1-\alpha_2)x}2f(x)(1-F(x)). \end{aligned}$$

or

$$\frac{d}{dx}\left(e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2\right) = -e^{-\theta(\alpha_1-\alpha_2)x}2f(x)(1-F(x)).$$

By the Fundamental Theorem of Calculus

$$e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2 = \int_0^x -e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds + C.$$

Since the LHS is zero at $x = \bar{x}$ then

$$C = \int_0^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds.$$

The unique solution $\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})$ therefore satisfies

$$e^{-2\theta\beta^{\theta}_{N-1}(x|\mathbf{z}_{N-2})}(1-F(x))^2 = \int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)s}2f(s)(1-F(s))ds.$$

Rearranging yields

$$\begin{aligned} \beta^{\theta}_{N-1}(x) &= -\frac{1}{2\theta}\ln\left(\frac{\int_x^{\bar{x}} e^{-\theta[(\alpha_1-\alpha_2)s]}2f(s)(1-F(s))ds}{(1-F(x))^2}\right) \\ &= -\frac{1}{2\theta}\ln\left(E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}\right]\right), \end{aligned}$$

which is $\beta^\theta_{N-1}(x)$, as given in Proposition 4.

Suppose $\beta^\theta_{t+1},\ldots,\beta^\theta_{N-1}$ are as given in Proposition 4 for rounds $t+1,\ldots,N-1$. Consider round t. The differential equation in the proof of Proposition 2 for round t is

$$\begin{aligned} & u'\left(\alpha_{N-t+1}x+\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum\nolimits_{j=1}^{t-1}\frac{1}{N-j}p_j\right)\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & -\left[\begin{array}{c} u\left(\alpha_{N-t}x+\beta^\theta_{t+1}(x)-\frac{1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j\right) \\ -u\left(\alpha_{N-t+1}x+\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j\right)\end{array}\right]\lambda^N_t(x), \end{aligned}$$

where we have used that $\beta^\theta_{t+1}(x)$ is independent of $\mathbf{z}_t$ by the induction hypothesis. We have

$$\begin{aligned} & \theta e^{-\theta[\alpha_{N-t+1}x+\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & -\left[e^{-\theta[\alpha_{N-t+1}x+\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}-e^{-\theta[\alpha_{N-t}x+\beta^\theta_{t+1}(x)-\frac{1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}\right]\lambda^N_t(x). \end{aligned}$$

Dividing both sides by $e^{-\theta[\alpha_{N-t+1}x-\frac{1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}$ yields

$$\begin{aligned} & -\theta e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & [e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}-e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}](N-t)\frac{f(x)}{1-F(x)}. \end{aligned}$$

Multiplying both sides by $\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}$ yields

$$\begin{aligned} & -\theta\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ = & [e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}-e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}](N-t+1)(1-F(x))^{N-t}f(x). \end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} & -\theta\frac{N-t+1}{N-t}(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}\beta^{\theta\prime}_t(x|\mathbf{z}_{t-1}) \\ & -e^{-\theta\frac{N-t+1}{N-t}\beta^\theta_t(x|\mathbf{z}_{t-1})}(N-t+1)(1-F(x))^{N-t}f(x) \\ = & -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta^\theta_{t+1}(x)]}(N-t+1)(1-F(x))^{N-t}f(x), \end{aligned}$$

i.e.,

$$\frac{d}{dx}((1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})}) = -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})x+\beta_{t+1}^\theta(x)]}(N-t+1)(1-F(x))^{N-t}f(x).$$

By the Fundamental Theorem of Calculus

$$\begin{aligned}&(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})}\\ = &\int_0^x -e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds + C.\end{aligned}$$

Since the LHS is zero at $x = \bar{x}$ then

$$C = \int_0^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds.$$

Hence the unique solution $\beta_t^\theta(x|\mathbf{z}_{t-1})$ satisfies

$$(1-F(x))^{N-t+1}e^{-\theta\frac{N-t+1}{N-t}\beta_t^\theta(x|\mathbf{z}_{t-1})} = \int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)(1-F(s))^{N-t}f(s)ds.$$

Thus

$$\begin{aligned}\beta_t^\theta(x) &= -\frac{N-t}{(N-t+1)\theta}\ln\left(\int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})s+\beta_{t+1}^\theta(s)]}(N-t+1)\frac{(1-F(s))^{N-t}}{(1-F(x))^{N-t+1}}f(s)ds\right)\\ &= -\frac{N-t}{(N-t+1)\theta}\ln\left\{E\left[e^{-\theta\left((\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^\theta(Z_t^{(N)})\right)}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right\},\end{aligned}$$

which establishes the result for round t and hence, by induction, the result for all t.

Next we establish that the bidding strategies are an equilibrium. It is sufficient to show that the following two-part claim holds for every t:

1. If $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$, i.e., it is optimal for a bidder with value x to bid $\beta_t^\theta(x)$ in round t.

2. If $x < z_{t-1}$ then $z_{t-1} \in \arg\max_y \pi_t(y, x|\mathbf{z}_{t-1})$.

The proof is by induction. Consider round $N-1$. Suppose that $x \geq z_{N-2}$. Consider an active bidder whose value is x and who bids y. Any bid below z_{N-2} is strictly dominated by a bid of z_{N-2} since both bids result in the same position while a bid of z_{N-2} yields higher compensation. Hence consider bids $y \geq z_{N-2}$.

With a bid of y the bidder wins Position 1 and obtains $\alpha_1 x - \beta^{\theta}_{N-1}(z_{N-1}) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j$ if $y > z_{N-1}$, and he obtains $\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j$ if $y < z_{N-1}$. Hence

$$\begin{aligned}\pi_{N-1}(y, x|\mathbf{z}_{N-2}) = & \int\limits_{z_{N-2}}^{y} u\left(\alpha_1 x - \beta^{\theta}_{N-1}(z_{N-1}) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} p_j\right) g^{(N-1)}_{N-1}(z_{N-1}|z_{N-2}) dz_{N-1} \\ & + \int\limits_{y}^{\bar{x}} u\left(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \tfrac{1}{N-j} p_j\right) g^{(N-1)}_{N-1}(z_{N-1}|z_{N-2}) dz_{N-1}.\end{aligned}$$

Differentiating with respect to y yields $\partial \pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y =$

$$\begin{array}{rc} & u(\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j) g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & u(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j) g^{(N-1)}_{N-1}(y|z_{N-2}) \\ + & u'(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j) \beta^{\theta\prime}_{N-1}(y)(1 - G^{(N-1)}_{N-1}(y|z_{N-2})). \end{array} \tag{3}$$

The necessary condition given in Proposition 2 for the general utility function u is

$$\begin{aligned} & u'\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum\nolimits_{j=1}^{N-2} \frac{1}{N-j} p_j\right) \beta^{\theta\prime}_{N-1}(y) \\ = & -\left[\begin{array}{c} u\left(\alpha_1 y - \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2} \frac{1}{N-j} p_j\right) \\ -u\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2} \frac{1}{N-j} p_j\right) \end{array}\right] \lambda^{N}_{N-1}(y). \end{aligned}$$

Substituting this expression into $\partial \pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y$ yields

$$\begin{array}{rc} & u(\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j) g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & u(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j) g^{(N-1)}_{N-1}(y|z_{N-2}) \\ - & \frac{u'(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j)}{u'(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2} \frac{1}{N-j} p_j)} \left[\begin{array}{c} u\left(\alpha_1 y - \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2} \frac{1}{N-j} p_j\right) \\ -u\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2} \frac{1}{N-j} p_j\right) \end{array}\right] g^{(N-1)}_{N-1}(y|z_{N-2}). \end{array}$$

This derivative has the same sign as

$$\begin{array}{ll} & u(\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j) - u(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j) \\ - & \frac{u'(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j)}{u'(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j)} \left[\begin{array}{c} u\left(\alpha_1 y - \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}p_j\right) \\ -u\left(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \sum_{j=1}^{N-2}\frac{1}{N-j}p_j\right) \end{array} \right]. \end{array}$$

Using that $u(x)$ has CARA we can write

$$\frac{u'(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j)}{u'(\alpha_2 y + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j)} = e^{-\theta\alpha_2(x-y)}.$$

We can write

$$u(\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j) - u(\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j)$$

as

$$\frac{e^{-\theta[\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]} - e^{-\theta[\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]}}{\theta}.$$

Hence the sign of the derivative is the same as the sign of

$$\begin{array}{l} e^{-\theta[\alpha_2 x + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]} - e^{-\theta[\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]} \\ -e^{-\theta\alpha_2(x-y)}\left(e^{-\theta[\alpha_2 y + \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]} - e^{-\theta[\alpha_1 y - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]}\right). \end{array}$$

We can rewrite this as

$$-e^{-\theta[\alpha_1 x - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]} + e^{-\theta\alpha_2(x-y)}e^{-\theta[\alpha_1 y - \beta^{\theta}_{N-1}(y) - \Sigma_{j=1}^{N-2}\frac{1}{N-j}p_j]}$$

which has the same sign as

$$-e^{-\theta\alpha_1 x} + e^{-\theta\alpha_2(x-y)}e^{-\theta\alpha_1 y}$$

which has the same sign as

$$-e^{-\theta\alpha_1(x-y)} + e^{-\theta\alpha_2(x-y)}.$$

Since $\alpha_1 > \alpha_2$, this expression is positive if $y < x$ and is negative if $y > x$. Thus $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y > 0$ if $y < x$ and $\partial\pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$ if $y > x$.

We have shown if $x \geq z_{N-2}$ then $x \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$, which establishes part 1 of the two-part claim. If $x < z_{N-2}$, then $y \geq z_{N-2}$ (since any bid below z_{N-2} is strictly dominated) implies $y \geq z_{N-2} > x$ and the above argument establishes bidding z_{N-2} is optimal since $\partial \pi_{N-1}(y, x|\mathbf{z}_{N-2})/\partial y < 0$ for all $y \geq z_{N-2}$, i.e., $z_{N-2} \in \arg\max_y \pi_{N-1}(y, x|\mathbf{z}_{N-2})$. This establishes part 2 of the two-part claim for round $N-1$.

Assume the two-part claim is true for rounds $t+1$ through $N-1$. We show it is true for round t. Let $\mathbf{z}_{t-1}$ be arbitrary. Suppose $x \geq z_{t-1}$. Consider an active bidder in the t-th round whose value is x and who bids as though his value is $y \geq z_{t-1}$. A bid below z_{t-1} is not optimal. We need to distinguish between two cases: (i) $y \in [z_{t-1}, x]$ and (ii) $y > x$, since his payoff function differs in each case. In what follows, we denote the payoff to a bid of y as $\pi_t^L(y, x|\mathbf{z}_{t-1})$ if $y \in [z_{t-1}, x]$ and as $\pi_t^H(y, x|\mathbf{z}_{t-1})$ if $y \geq x$.

Case (i): Consider a bid $y \in [z_{t-1}, x]$. If $z_t \in [z_{t-1}, y]$ the bidder continues to round $t+1$ where, by the induction hypothesis, he optimally bids x and he has an expected payoff of $\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \geq y$ he obtains a payoff of $\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1} \frac{1}{N-j} p_j$ in round t. Hence his payoff is

$$\begin{aligned}\pi_t^L(y, x|\mathbf{z}_{t-1}) = \quad & \textstyle\int_{z_{t-1}}^{y} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t) g_t^{(N-1)}(z_t|z_{t-1}) dz_t \\ & + \textstyle\int_{y}^{\bar{x}} u\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1} \frac{1}{N-j} p_j\right) g_t^{(N-1)}(z_t|z_{t-1}) dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - u\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1} \tfrac{1}{N-j} p_j\right)] g_t^{(N-1)}(y|z_{t-1}) \\ & + u'\left(\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1} \tfrac{1}{N-j} p_j\right) \beta_t^{\theta\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Rewriting

$$\begin{aligned}\frac{\partial \pi_t^L(y, x|\mathbf{z}_{t-1})}{\partial y} = \quad & [\Pi_{t+1}(x|\mathbf{z}_{t-1}, y) - \tfrac{1 - e^{-\theta[\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1} \frac{1}{N-j} p_j]}}{\theta}] g_t^{(N-1)}(y|z_{t-1}) \\ & + e^{-\theta[\alpha_{N-t+1}x + \beta_t^\theta(y) - \Sigma_{j=1}^{t-1} \frac{1}{N-j} p_j]} \beta_t^{\theta\prime}(y)(1 - G_t^{(N-1)}(y|z_{t-1})).\end{aligned}$$

Using the expression for $\beta_t^{\theta\prime}(y)$ from the necessary condition for equilibrium from Proposition 2 for round t and substituting yields

$$\begin{aligned}\frac{\partial \pi_t^L(y,x|\mathbf{z}_{t-1})}{\partial y} &= [\Pi_{t+1}(x|\mathbf{z}_{t-1},y) - \tfrac{1-e^{-\theta[\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]}}{\theta}]g_t^{(N-1)}(y|z_{t-1}) \\ &\quad -e^{-\theta\alpha_{N-t+1}(x-y)}\frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t+1}y+\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]} \\ -e^{-\theta[\alpha_{N-t}y+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}\end{array}\right]g_t^{(N-1)}(y|z_{t-1}).\end{aligned}$$

Simplifying yields $\partial \pi_t^L(y,x|\mathbf{z}_{t-1})/\partial y$ as

$$\left(\Pi_{t+1}(x|\mathbf{z}_{t-1},y) - \frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}\right]\right)g_t^{(N-1)}(y|z_{t-1}).$$

We show that $\partial \pi_t^L(y,x|\mathbf{z}_{t-1})/\partial y > 0$ for $y < x$. If the bid at round t is y, then $\Pi_{t+1}(x|\mathbf{z}_{t-1},y)$ is the equilibrium payoff at round $t+1$ of a bidder with value x. If he were to deviate from equilibrium and bid y at round $t+1$, then he obtains position $N-t$ (since y is the smallest value of a rival bidder) and he receives $\beta_{t+1}^\theta(y)$ at round $t+1$ and pays $\frac{1}{N-t}\beta_t^\theta(y)+\sum_{j=1}^{t-1}\frac{1}{N-j}p_j$. By the induction hypothesis, this payoff is less than his equilibrium payoff, i.e.,

$$\Pi_{t+1}(x|\mathbf{z}_{t-1},y) > \frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{1}{N-j}p_j]}\right].$$

Since $\alpha_{N-t} > \alpha_{N-t+1}$ and $x > y$ we have

$$\alpha_{N-t}x > \alpha_{N-t}y + \alpha_{N-t+1}(x-y)$$

and hence

$$\begin{aligned}&\frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{p_j}{N-j}]}\right] \\ > \;&\frac{1}{\theta}\left[1-e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\sum_{j=1}^{t-1}\frac{p_j}{N-j}]}\right].\end{aligned}$$

Thus $\Pi_{t+1}(x|\mathbf{z}_{t-1},y)$ is greater than the RHS of this inequality and hence $\partial \pi_t^L(y,x|\mathbf{z}_{t-1})/\partial y > 0$ for $y < x$.

Case (ii): Consider a bid $y \geq x$. If $z_t \in [z_{t-1}, x]$, then the bidder continues to round $t+1$ and, by part 1 of induction hypothesis, he bids x and obtains

$\Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)$. If $z_t \in [x, y]$, then he continues to round $t+1$ and, by part 2 of the induction hypothesis, he bids z_t and obtains a payoff of

$$\alpha_{N-t}x + \beta^{\theta}_{t+1}(z_t) - \frac{1}{N-t}\beta^{\theta}_t(z_t) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j.$$

If $z_t > y$ then in round t he obtains position $N-t+1$ and his payoff is $\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j$. Thus his expected payoff at round t is

$$\begin{aligned}\pi^H_t(y, x|\mathbf{z}_{t-1}) = \quad & \int^x_{z_{t-1}} \Pi_{t+1}(x|\mathbf{z}_{t-1}, z_t)g^{(N-1)}_t(z_t|z_{t-1})dz_t \\ & + \int^y_x u(\alpha_{N-t}x + \beta^{\theta}_{t+1}(z_t) - \tfrac{1}{N-t}\beta^{\theta}_t(z_t) - \Sigma^{t-1}_{j=1}\tfrac{1}{N-j}p_j)g^{(N-1)}_t(z_t|z_{t-1})dz_t, \\ & + \int^{\bar{x}}_y u\left(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\tfrac{1}{N-j}p_j\right) g^{(N-1)}_t(z_t|z_{t-1})dz_t.\end{aligned}$$

Differentiating with respect to y yields

$$\begin{aligned}\frac{\partial\pi^H_t(y, x|\mathbf{z}_{t-1})}{\partial y} \quad = \quad & \left[\begin{array}{c} u\left(\alpha_{N-t}x + \beta^{\theta}_{t+1}(y) - \frac{1}{N-t}\beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j\right) \\ -u\left(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j\right)\end{array}\right] g^{(N-1)}_t(y|z_{t-1}) \\ & +u'\left(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j\right)\beta^{\theta\prime}_t(y)(1 - G^{(N-1)}_t(y|z_{t-1})).\end{aligned}$$

Using the expression for $\beta^{\theta\prime}_t(y)$ from the necessary condition for equilibrium from Proposition 2 for round t and substituting gives $\partial\pi^H_t(y, x|\mathbf{z}_{t-1})/\partial y$ as

$$\begin{aligned}& \left[\begin{array}{c} u(\alpha_{N-t}x + \beta^{\theta}_{t+1}(y) - \frac{1}{N-t}\beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j) \\ -u(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j)\end{array}\right] g^{(N-1)}_t(y|z_{t-1}) \\ & -\frac{u'(\alpha_{N-t+1}x + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j)}{u'(\alpha_{N-t+1}y + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j)} \\ & \times\left[\begin{array}{c} u(\alpha_{N-t}y + \beta^{\theta}_{t+1}(y) - \frac{1}{N-t}\beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j) \\ -u(\alpha_{N-t+1}y + \beta^{\theta}_t(y) - \Sigma^{t-1}_{j=1}\frac{1}{N-j}p_j)\end{array}\right] g^{(N-1)}_t(y|z_{t-1}).\end{aligned}$$

Since bidders have CARA preferences, then $\partial \pi_t^H(y,x|\mathbf{z}_{t-1})/\partial y$ is

$$\begin{aligned}
& \frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t+1}x+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]} \\ -e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]} \end{array}\right] g_t^{(N-1)}(y|z_{t-1}) \\
& -e^{-\theta\alpha_{N-t+1}(x-y)}\frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t+1}y+\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]} \\ -e^{-\theta[\alpha_{N-t}y+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]} \end{array}\right] g_t^{(N-1)}(y|z_{t-1}). \\
= & \frac{1}{\theta}\left[\begin{array}{c} e^{-\theta[\alpha_{N-t}y+\alpha_{N-t+1}(x-y)+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]} \\ -e^{-\theta[\alpha_{N-t}x+\beta_{t+1}^\theta(y)-\frac{1}{N-t}\beta_t^\theta(y)-\Sigma_{j=1}^{t-1}\frac{1}{N-j}p_j]} \end{array}\right] g_t^{(N-1)}(y|z_{t-1}).
\end{aligned}$$

This has the same sign as

$$e^{-\theta\alpha_{N-t+1}(x-y)} - e^{-\theta\alpha_{N-t}(x-y)},$$

which is negative since $\alpha_{N-t} > \alpha_{N-t+1}$ and $y > x$.

We have shown if $x \geq z_{t-1}$ then $x \in \arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$. If $x < z_{t-1}$, then $y \geq z_{t-1}$ (since any bid below z_{t-1} is strictly dominated) implies $y \geq z_{t-1} > x$ and the above argument establishes bidding z_{t-1} is optimal since $\partial\pi_t(y,x|\mathbf{z}_{t-1})/\partial y < 0$ for all $y \geq z_{t-1}$, i.e., $z_{t-1} \in \arg\max_y \pi_t(y,x|\mathbf{z}_{t-1})$. This establishes the two-part claim for round t, and completes the proof by induction. □

Proof of Proposition 5: We first show that for each t we have that $\beta_t^0(x) > \beta_t^\theta(x)$ for $\theta > 0$ and $x < \bar{x}$. Consider round $t = N-1$. Since e^{-x} is convex, Jensen's inequality implies

$$e^{-E\left[\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)}>x>Z_{N-2}^{(N)}\right]} < E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right].$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\begin{aligned}
\beta_{N-1}^0(x) &= \frac{1}{2}E\left[(\alpha_1-\alpha_2)Z_{N-1}^{(N)}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right] \\
&> -\frac{1}{2\theta}\ln\left\{E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right\} \\
&= \beta_{N-1}^\theta(x).
\end{aligned}$$

Assume that $\beta^0_{t+1}(x) > \beta^\theta_{t+1}(x)$ for $x < \bar{x}$. We show that $\beta^0_t(x) > \beta^\theta_t(x)$ for $x < \bar{x}$. For $z < \bar{x}$ we have

$$(\alpha_{N-t} - \alpha_{N-t+1})\, z + \beta^0_{t+1}(z) > (\alpha_{N-t} - \alpha_{N-t+1})\, z + \beta^\theta_{t+1}(z).$$

Multiplying through by $-\theta$ and applying the exponential function to both sides gives

$$e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta^0_{t+1}(z)\right]} < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta^\theta_{t+1}(z)\right]}.$$

Hence

$$\begin{aligned} & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_t+\beta^0_{t+1}(Z^{(N)}_t)\right]}|Z^{(N)}_t > x > Z^{(N)}_{t-1}\right] \\ < \ & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_t+\beta^\theta_{t+1}(Z^{(N)}_t)\right]}|Z^{(N)}_t > x > Z^{(N)}_{t-1}\right]. \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} & e^{-\theta E\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_t+\beta^0_{t+1}(Z^{(N)}_t)|Z^{(N)}_t>x>Z^{(N)}_{t-1}\right]} \\ < \ & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_t+\beta^0_{t+1}(Z^{(N)}_t)\right]}|Z^{(N)}_t > x > Z^{(N)}_{t-1}\right], \end{aligned}$$

and thus

$$\begin{aligned} & e^{-\theta E\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_t+\beta^0_{t+1}(Z^{(N)}_t)|Z^{(N)}_t>x>Z^{(N)}_{t-1}\right]} \\ < \ & E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_t+\beta^\theta_{t+1}(Z^{(N)}_t)\right]}|Z^{(N)}_t > x > Z^{(N)}_{t-1}\right]. \end{aligned}$$

Taking the log of both sides and then multiplying both sides by $-(N - t)/((N-t+1)\,\theta)$ yields $\beta^0_t(x) > \beta^\theta_t(x)$. We have shown for each t that $\beta^0_t(x) > \beta^\theta_t(x)$ for $\theta > 0$ and $x < \bar{x}$.

Next we show that for each t we have that $\beta^\theta_t(x) > \underline{\beta}_t(x)$ for $\theta > 0$ and $x < \bar{x}$. Consider $t = N-1$. For $z > x$ we have

$$e^{-\theta(\alpha_1-\alpha_2)z} < e^{-\theta(\alpha_1-\alpha_2)x},$$

and hence

$$E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}\right] < e^{-\theta(\alpha_1-\alpha_2)x}.$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\beta^{\theta}_{N-1}(x) = -\frac{1}{2\theta}\ln\left\{E\left[e^{-\theta(\alpha_1-\alpha_2)Z^{(N)}_{N-1}}|Z^{(N)}_{N-1} > x > Z^{(N)}_{N-2}\right]\right\} > \frac{\alpha_1-\alpha_2}{2}x = \underline{\beta}_{N-1}(x).$$

Assume that $\beta^{\theta}_{t+1}(x) > \underline{\beta}_{t+1}(x)$ for $x < \bar{x}$. We show that $\beta^{\theta}_{t}(x) > \underline{\beta}_{t}(x)$ for $x < \bar{x}$. For $z > x$ we have

$$e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta^{\theta}_{t+1}(z)\right]} < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\underline{\beta}_{t+1}(x)\right]},$$

and hence

$$E\left[e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_{t}+\beta^{\theta}_{t+1}(Z^{(N)}_{t})\right]}|Z^{(N)}_{t} > x > Z^{(N)}_{t-1}\right] < e^{-\theta\left[(\alpha_{N-t}-\alpha_{N-t+1})x+\underline{\beta}_{t+1}(x)\right]}.$$

By the analogous argument as above, we obtain

$$\begin{aligned}\beta^{\theta}_{t}(x) &= -\frac{N-t}{(N-t+1)\,\theta}\ln\left\{E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z^{(N)}_{t}+\beta^{\theta}_{t+1}(Z^{(N)}_{t})]}|Z^{(N)}_{t} > x > Z^{(N)}_{t-1}\right]\right\}\\ &> \frac{N-t}{N-t+1}\left[(\alpha_{N-t}-\alpha_{N-t+1})\,x+\underline{\beta}_{t+1}(x)\right]\\ &= \underline{\beta}_{t}(x),\end{aligned}$$

where the last equality follows since

$$\begin{aligned}&\frac{N-t}{N-t+1}\left[(\alpha_{N-t}-\alpha_{N-t+1})\,x+\underline{\beta}_{t+1}(x)\right]\\ =\;& \frac{N-t}{N-t+1}(\alpha_{N-t}-\alpha_{N-t+1})\,x + \frac{N-t}{N-t+1}\left(\sum_{m=1}^{N-t-1}\frac{1}{N-t}\alpha_m - \frac{N-t-1}{N-t}\alpha_{N-t}\right)x\\ =\;& \left(\sum_{m=1}^{N-t-1}\frac{1}{N-t+1}\alpha_m - \frac{N-t-1}{N-t+1}\alpha_{N-t} + \frac{N-t}{N-t+1}(\alpha_{N-t}-\alpha_{N-t+1})\right)x\\ =\;& \left(\sum_{m=1}^{N-t}\frac{1}{N-t+1}\alpha_m - \frac{N-t}{N-t+1}\alpha_{N-t+1}\right)x\\ =\;& \underline{\beta}_{t}(x).\end{aligned}$$

□

Proof of Proposition 6: We first show that for each t we have $\beta_t^{\theta'}(x) < \beta_t^{\theta}(x)$ for $\theta' > \theta$ and $x < \bar{x}$. Consider round $t = N-1$. Since $f(s) = s^{\frac{\theta}{\theta'}}$ is concave, by Jensen's inequality we have

$$\left(E\left[e^{-\theta'(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right)^{\frac{\theta}{\theta'}} > E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right].$$

Taking the log of both sides and then dividing both sides by -2θ yields

$$\begin{aligned}\beta_{N-1}^{\theta}(x) &= -\frac{1}{2\theta}\ln\left(E\left[e^{-\theta(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right) \\ &> -\frac{1}{2\theta'}\ln\left(E\left[e^{-\theta'(\alpha_1-\alpha_2)Z_{N-1}^{(N)}}|Z_{N-1}^{(N)} > x > Z_{N-2}^{(N)}\right]\right) \\ &= \beta_{N-1}^{\theta'}(x).\end{aligned}$$

Assume that $\beta_{t+1}^{\theta'}(x) < \beta_{t+1}^{\theta}(x)$ for $x < \bar{x}$. We show that $\beta_t^{\theta'}(x) < \beta_t^{\theta}(x)$ for $x < \bar{x}$. By Jensen's inequality we have

$$\begin{aligned}& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\ < \; & \left(E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right)^{\frac{\theta}{\theta'}}\end{aligned}$$

and since $\beta_{t+1}^{\theta'}(x) < \beta_{t+1}^{\theta}(x)$ then

$$\begin{aligned}& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\ < \; & E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right].\end{aligned}$$

Hence

$$\begin{aligned}& E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right] \\ < \; & \left(E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right)^{\frac{\theta}{\theta'}}.\end{aligned}$$

Taking the log of both sides and multiplying both sides by $-(N-t)/((N-t+1)\,\theta)$

yields

$$\begin{aligned}
\beta_t^{\theta'}(x) &= -\frac{N-t}{(N-t+1)\,\theta'} \ln\left\{E\left[e^{-\theta'[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta'}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right\} \\
&< -\frac{N-t}{(N-t+1)\,\theta} \ln\left\{E\left[e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})Z_t^{(N)}+\beta_{t+1}^{\theta}(Z_t^{(N)})]}|Z_t^{(N)} > x > Z_{t-1}^{(N)}\right]\right\} \\
&= \beta_t^{\theta}(x).
\end{aligned}$$

Next we show that for each t we have $\lim_{\theta\to\infty}\beta_t^{\theta}(x) = \underline{\beta}_t(x)$ for all x. For $t = N-1$ the limit is obtained directly. Specifically, after applying l'Hopital's rule, we see that

$$\lim_{\theta\to\infty}\beta_{N-1}^{\theta}(x) = \frac{1}{2}\left(\alpha_1-\alpha_2\right)\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}$$

where $g_{N-1}^{(N)}(z|x) = 2f(z)(1-F(z))/(1-F(x))^2$. Van Essen and Wooders (2016, p. 239) established that

$$\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta z}g_{N-1}^{(N)}(z|x)dz} = x,$$

which implies that

$$\lim_{\theta\to\infty}\frac{\int_x^{\bar{x}} ze^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz}{\int_x^{\bar{x}} e^{-\theta(\alpha_1-\alpha_2)z}g_{N-1}^{(N)}(z|x)dz} = x,$$

Hence,

$$\lim_{\theta\to\infty}\beta_{N-1}^{\theta}(x) = \frac{1}{2}\left(\alpha_1-\alpha_2\right)x = \underline{\beta}_{N-1}(x).$$

Observe that $\beta_{N-1}^{\theta}(x)$ is continuous in x on the compact set $[0,\bar{x}]$ for each θ, it converges pointwise to $\underline{\beta}_{N-1}(x)$, which is continuous on $[0,\bar{x}]$, and it is decreasing in θ. Hence β_{N-1}^{θ} converges uniformly to $\underline{\beta}_{N-1}$ on $[0,\bar{x}]$ by Theorem 7.12 of Rudin (1976).

Assume that $\beta_{t+1}^{\theta}(x)$ converges uniformly to $\underline{\beta}_{t+1}(x)$ on $[0,\bar{x}]$. We show that $\beta_t^{\theta}(x)$ converges uniformly to $\underline{\beta}_t(x)$. The CARA bid function in round t is

$$\beta_t^{\theta}(x) = -\frac{N-t}{(N-t+1)\,\theta}\ln\left(\int_x^{\bar{x}} e^{-\theta[(\alpha_{N-t}-\alpha_{N-t+1})z+\beta_{t+1}^{\theta}(z)]}g_t^{(N)}(z|x)dz\right).$$

Let $\Delta > 0$ be arbitrary. Since $\beta_t^\theta(x)$ is decreasing in θ and since $\beta_{t+1}^\theta \to \underline{\beta}_{t+1}$ uniformly as $\theta \to \infty$, then there is a $\bar{\theta}$ such that for all $\theta \geq \bar{\theta}$ we have

$$\beta_{t+1}^\theta(x) \leq \sum_{m=1}^{N-t-1} \frac{m}{N-t} \left(\alpha_m - \alpha_{m+1}\right) x + \Delta$$

for $x \in [0, \bar{x}]$. Define

$$\bar{\beta}_t^\theta(x) \equiv -\frac{N-t}{(N-t+1)\,\theta} \ln \left(\int_x^{\bar{x}} e^{-\theta[z(\alpha_{N-t}-\alpha_{N-t+1}+\sum_{m=1}^{N-t-1} \frac{m}{N-t}(\alpha_m-\alpha_{m+1}))+\Delta]} g_t^{(N)}(z|x) \right) dz.$$

Then $\beta_t^\theta(x) \leq \bar{\beta}_t^\theta(x)$ for $\theta \geq \bar{\theta}$ and $x \in [0, \bar{x}]$. By Proposition 5 we have $\underline{\beta}_t(x) \leq \beta_t^\theta(x)$ and thus

$$\underline{\beta}_t(x) \leq \beta_t^\theta(x) \leq \bar{\beta}_t^\theta(x)$$

for $\theta \geq \bar{\theta}$ and $x \in [0, \bar{x}]$.

We establish that $\beta_t^\theta(x)$ converges pointwise to $\underline{\beta}_t(x)$ for each $x \in [0, \bar{x}]$. Define

$$C = \alpha_{N-t} - \alpha_{N-t+1} + \sum_{m=1}^{N-t-1} \frac{m}{N-t} \left(\alpha_m - \alpha_{m+1}\right).$$

Applying L'Hopital's rule and using the same argument as for round $N-1$, we have

$$\begin{aligned}
\lim_{\theta\to\infty} \bar{\beta}_t^\theta(x) &= \frac{N-t}{N-t+1} \lim_{\theta\to\infty} \frac{\int_x^{\bar{x}} (zC+\Delta) e^{-\theta(zC+\Delta)} g_t^{(N)}(z|x) dz}{\int_x^{\bar{x}} e^{-\theta(zC+\Delta)} g_t^{(N)}(z|x) dz} \\
&= \frac{N-t}{N-t+1} \left(C \lim_{\theta\to\infty} \frac{\int_x^{\bar{x}} z e^{-\theta z C} g_t^{(N)}(z|x) dz}{\int_x^{\bar{x}} e^{-\theta z C} g_t^{(N)}(z|x) dz} + \Delta \right) \\
&= \frac{N-t}{N-t+1} \left(Cx + \Delta \right),
\end{aligned}$$

where the last inequality holds by Van Essen and Wooders (2016). Substituting for C and simplifying yields

$$\lim_{\theta\to\infty} \bar{\beta}_t^\theta(x) = \underline{\beta}_t(x) + \frac{N-t}{N-t+1} \Delta.$$

Since the inequality

$$\underline{\beta}_t(x) \leq \lim_{\theta\to\infty} \beta_t^\theta(x) \leq \lim_{\theta\to\infty} \bar{\beta}_t^\theta(x) = \underline{\beta}_t(x) + \frac{N-t}{N-t+1}\Delta$$

holds for arbitrary $\Delta > 0$, it follows that $\lim_{\theta\to\infty} \beta_t^\theta(x) = \underline{\beta}_t(x)$. By the same argument as for β_{N-1}^θ, we have that β_t^θ converges uniformly to $\underline{\beta}_t$ on $[0, \bar{x}]$. $\square$

Proof of Proposition 7: We first show following $\underline{\beta}$ guarantees a bidder with value x a payoff at round t of at least

$$\bar{v}_t(x; \mathbf{p}_{t-1}) = \left(\sum_{m=1}^{N-t+1} \frac{\alpha_m}{N-t+1}\right) x - \sum_{m=1}^{t-1} \frac{1}{N-m} p_m,$$

when $\mathbf{p}_{t-1}$ is the sequence of dropout prices at prior rounds.

Consider round $N-1$. A bidder with value x whose dropout price is b either (i) drops at b and obtains a payoff of $\alpha_2 x + b - \sum_{m=1}^{N-2} \frac{1}{N-m} p_m$ or (ii) his rival drops first at $p_{N-1} \leq b$ and he obtains a payoff of $\alpha_1 x - p_{N-1} - \sum_{m=1}^{N-2} \frac{1}{N-m} p_m$. In the second case, his payoff is at least $\alpha_1 x - b - \sum_{m=1}^{N-2} \frac{1}{N-m} p_m$. The bidder maximizes his minimum payoff when b satisfies

$$\alpha_2 x + b - \sum_{m=1}^{N-2} \frac{1}{N-m} p_m = \alpha_1 x - b - \sum_{m=1}^{N-2} \frac{1}{N-m} p_m,$$

i.e., $b = \frac{1}{2}(\alpha_1 - \alpha_2)x$. Hence at round $N-1$ the bidder guarantees himself a payoff of at least

$$\bar{v}_{N-1}(x; \mathbf{p}_{N-2}) = \frac{\alpha_1 + \alpha_2}{2} x - \sum_{m=1}^{N-2} \frac{1}{N-m} p_m$$

following $\underline{\beta}_{N-1}(x) = \frac{1}{2}(\alpha_1 - a_2)x$.

Suppose that at round $t+1$, given dropout prices $\mathbf{p}_t$, a bidder with value x can guarantee himself at least

$$\bar{v}_{t+1}(x; \mathbf{p}_t) = \sum_{m=1}^{N-t} \frac{\alpha_m}{N-t} x - \sum_{m=1}^{t} \frac{1}{N-m} p_m,$$

by following

$$\underline{\beta}_s(x) = \left(\sum_{m=1}^{N-s} \frac{1}{N-s+1} \alpha_m - \frac{N-s}{N-s+1} \alpha_{N-s+1} \right) x$$

for $s = t+1, \ldots, N-1$. We show that at round t he can guarantee himself at least

$$\bar{v}_t(x; \mathbf{p}_{t-1}) = \sum_{m=1}^{N-t+1} \frac{\alpha_m}{N-t+1} x - \sum_{m=1}^{t-1} \frac{1}{N-m} p_m,$$

by following

$$\underline{\beta}_s(x) = \left(\sum_{m=1}^{N-s} \frac{1}{N-s+1} \alpha_m - \frac{N-s}{N-s+1} \alpha_{N-s+1} \right) x$$

for $s = t, \ldots, N-1$.

A bidder with value x whose dropout price is b at round t either (i) drops at b and obtains a payoff of $\alpha_{N-t+1} x + b - \sum_{m=1}^{t-1} \frac{1}{N-m} p_m$ or (ii) one of his rivals drops first at $p_t \leq b$ and, by induction, he obtains at least

$$\begin{aligned} \bar{v}_{t+1}(x; \mathbf{p}_{t-1}, p_t) &= \sum_{m=1}^{N-t} \frac{\alpha_m}{N-t} x - \frac{1}{N-t} p_t - \sum_{m=1}^{t-1} \frac{1}{N-m} p_m \\ &\geq \sum_{m=1}^{N-t} \frac{\alpha_m}{N-t} x - \frac{1}{N-t} b - \sum_{m=1}^{t-1} \frac{1}{N-m} p_m. \end{aligned}$$

The bidder maximizes his minimum payoff when b satisfies

$$\alpha_{N-t+1} x + b = \sum_{m=1}^{N-t} \frac{\alpha_m}{N-t} x - \frac{1}{N-t} b,$$

i.e.,

$$b = \left(\sum_{m=1}^{N-t} \frac{1}{N-t+1} \alpha_m - \frac{N-t}{N-t+1} \alpha_{N-t+1} \right) x = \underline{\beta}_t(x).$$

Substituting b into $\alpha_{N-t+1} x + b - \sum_{m=1}^{t-1} \frac{1}{N-m} p_m$ and simplifying shows that his payoff is at least $\bar{v}_t(x; \mathbf{p}_{t-1})$.

Next, we show that $\bar{v}_t(x;\mathbf{p}_{t-1})$ is the largest payoff a bidder with value x can guarantee at round t given dropout prices $\mathbf{p}_{t-1}$. Suppose to the contrary he can guarantee himself $v'_t > \bar{v}_t(x;\mathbf{p}_{t-1})$. If all active bidder have the same value x then, since the game is symmetric, each such bidder can guarantee himself at least v'_t and hence the total guaranteed payoffs of the active bidders is at least

$$\begin{aligned}(N-t+1)v'_t &> (N-t+1)\left(\sum_{m=1}^{N-t+1}\frac{\alpha_m}{N-t+1}x-\sum_{m=1}^{t-1}\frac{1}{N-m}p_m\right)\\ &= \sum_{m=1}^{N-t+1}\alpha_m x-(N-t+1)\sum_{m=1}^{t-1}\frac{1}{N-m}p_m,\end{aligned}$$

which is a contraction since the RHS is the total surplus that can be obtained by the active bidders at round t. The first term is the surplus realized from allocating positions 1 through $N-t+1$ to the active bidders, and the second term is the compensation they owe.

We have established that $\underline{\beta}$ is a maxmin perfect strategy. Next we show that $\underline{\beta}$ is the unique maxmin perfect strategy. As a first step, we establish at each round t that a bidder with value x can be held to a payoff $\bar{v}_t(x;\mathbf{p}_{t-1})$ given drop out prices $\mathbf{p}_{t-1}$.

Consider a bidder with value x at round $N-1$ with dropout prices $\mathbf{p}_{N-2}$. Suppose his rival bids $\underline{\beta}_{N-1}(x)$. If the bidder bids $b<\underline{\beta}_{N-1}(x)$, then his payoff is

$$\alpha_2 x+b-\sum_{m=1}^{N-2}\frac{p_m}{N-m}<\alpha_2 x+\underline{\beta}_{N-1}(x)-\sum_{m=1}^{N-2}\frac{p_m}{N-m}=\bar{v}_{N-1}(x;\mathbf{p}_{N-2}).$$

If he bids $b>\underline{\beta}_{N-1}(x)$ then his payoff is

$$\alpha_1 x-\underline{\beta}_{N-1}(x)-\sum_{m=1}^{N-2}\frac{p_m}{N-m}=\bar{v}_{N-1}(x;\mathbf{p}_{N-2}).$$

In both case, his payoff is at most $\bar{v}_{N-1}(x;\mathbf{p}_{N-2})$, which establishes he is held to $\bar{v}_{N-1}(x;\mathbf{p}_{N-2})$.

Suppose the claim is true for rounds $t+1, \ldots, N-1$. We show it holds for round t. Consider a bidder with value x at round t with dropout prices $\mathbf{p}_{t-1}$. Suppose at each round $s = t, \ldots, N-1$ that each of his rivals bids $\underline{\beta}_s(x)$ at round s given drop out prices $\mathbf{p}_{s-1}$. If at round t the bidder bids $b < \underline{\beta}_t(x)$ his payoff is

$$\alpha_{N-t+1}x + b - \sum_{m=1}^{t-1} \frac{p_m}{N-m} < \alpha_{N-t+1}x + \underline{\beta}_t(x) - \sum_{m=1}^{t-1} \frac{p_m}{N-m} = \bar{v}_t(x; \mathbf{p}_{t-1}).$$

If he bids $b > \underline{\beta}_t(x)$, then he continues to round $t+1$ and by the induction hypothesis his rivals hold him to $\bar{v}_{t+1}(x; \mathbf{p}_{t-1}, \underline{\beta}_t(x))$. Straight forward algebra establishes that

$$\bar{v}_{t+1}(x; \mathbf{p}_{t-1}, \underline{\beta}_t(x)) = \bar{v}_t(x; \mathbf{p}_{t-1}).$$

This establishes the claim holds for all rounds.

Finally, we show that $\underline{\beta}$ is the unique maxmin perfect strategy. Suppose that there is another maxmin strategy $\hat{\beta} \neq \underline{\beta}$. Then for some x, t, and $\mathbf{p}_{t-1}$ we have that $\hat{\beta}_t(x; \mathbf{p}_{t-1}) \neq \underline{\beta}_t(x)$. Consider a bidder with value x at round t, given dropout prices $\mathbf{p}_{t-1}$, who follows $\hat{\beta}$. Suppose that at each round $s = t, \ldots, N-1$ that his rivals bid $\underline{\beta}_s(x)$ at round s. If $\hat{\beta}_t(x; \mathbf{p}_{t-1}) < \underline{\beta}_t(x)$ then bidder i drops out at round t and obtains the payoff

$$\alpha_{N-t+1}x + \hat{\beta}_t(x; \mathbf{p}_{t-1}) - \sum_{m=1}^{t-1} \frac{p_m}{N-m} < \alpha_{N-t+1}x + \underline{\beta}_t(x) - \sum_{m=1}^{t-1} \frac{p_m}{N-m} = \bar{v}_t(x; \mathbf{p}_{t-1}).$$

If $\hat{\beta}_t(x; \mathbf{p}_{t-1}) > \underline{\beta}_t(x)$ and his rivals bid $(\hat{\beta}_t(x; \mathbf{p}_{t-1}) + \underline{\beta}_t(x))/2$ at round t and bids $\underline{\beta}_s(x)$ at each round $s = t+1, \ldots, N-1$ then the bidder's payoff at round t is at most

$$\bar{v}_{t+1}(x; \mathbf{p}_{t-1}, \frac{1}{2}(\hat{\beta}_t(x; \mathbf{p}_{t-1}) + \underline{\beta}_t(x)))$$

by the immediately prior claim. Since $\bar{v}_{t+1}(x; \mathbf{p}_{t-1}, p_t)$ is decreasing in p_t we have

$$\bar{v}_{t+1}(x; \mathbf{p}_{t-1}, \frac{1}{2}(\hat{\beta}_t(x; \mathbf{p}_{t-1}) + \underline{\beta}_t(x))) < \bar{v}_{t+1}(x; \mathbf{p}_{t-1}, \underline{\beta}_t(x)) = \bar{v}_t(x; \mathbf{p}_{t-1}),$$

which contradicts that $\hat{\beta}$ is a security strategy. □

Proof of Proposition 8: Consider a bidder with value x who follows $\underline{\beta}$. If he drops out at round t his payoff is

$$\alpha_{N-t+1}x+\underline{\beta}_t(x)-\sum_{m=1}^{t-1}\frac{p_m}{N-m}=\left(\sum_{m=1}^{N-t+1}\frac{\alpha_m}{N-t+1}\right)x-\sum_{m=1}^{t-1}\frac{p_m}{N-m}=\bar{v}_t(x;\mathbf{p}_{t-1}),$$

when $\mathbf{p}_{t-1}$ is the sequence of drop out prices in prior rounds. In other words, a bidder who follows the security strategy obtains exactly his security payoff if he drops.

Order the bidders so that $x_1>\ldots>x_N$. When all the bidders follow $\underline{\beta}$, since $\underline{\beta}_t$ is increasing for each t, then bidder N drops in round 1 and wins position N, bidder $N-1$ drops in round 2 and wins position $N-1$, and so on. Bidder i drops in round $N-i+1$, wins position i, and obtains

$$\bar{v}_{N-i+1}(x_i;\mathbf{p}_{t-1})=\left(\sum_{m=1}^{i}\frac{\alpha_m}{i}\right)x_i-\sum_{j=1}^{N-i}\frac{p_j}{N-j}.$$

Reversing the order of the terms in the second sum by reindexing and setting $j=N-(i+m)+1$, we can write

$$\bar{v}_{N-i+1}(x_i;\mathbf{p}_{t-1})=\left(\sum_{m=1}^{i}\frac{\alpha_m}{i}\right)x_i-\sum_{m=1}^{N-i}\frac{p_{N-(i+m)+1}}{i+m-1}.$$

Since all the bidders follow $\underline{\beta}$ then bidder $i+m$ wins position $i+m$ in round $N-(i+m)+1$ and the dropout price is

$$p_{N-(i+m)+1}=\underline{\beta}_{N-(i+m)+1}(x_{i+m})=\sum_{r=1}^{i+m-1}\frac{r}{i+m}\left(\alpha_r-\alpha_{r+1}\right)x_{i+m}.$$

Substituting yields that the payoff of bidder i is

$$\bar{v}_{N-i+1}(x_i;\mathbf{p}_{t-1})=\frac{1}{i}\left(\sum_{m=1}^{i}\alpha_m\right)x_i-\sum_{m=1}^{N-i}\frac{1}{i+m-1}\left[\sum_{r=1}^{i+m-1}\frac{r}{i+m}\left(\alpha_r-\alpha_{r+1}\right)x_{i+m}\right],$$

which from Proposition 1 is his Shapley value payoff. □

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